Coleman Theory (Starts on page 4)

Review : (1) p-adic wt space $\qquad \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$

fix $p>5$ a prime integer

Classical weights (elliptic modular forms) $\in \mathbb{Z}$

Space of p-adic weights $= W$ (think as analytic space attached to the Iwasawa algebra $\Lambda = \varprojlim \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \simeq \mathbb{Z}_p[\![\mu_{p-1} \times (1+p\mathbb{Z}_p)]\!]$

$$\simeq \mathbb{Z}_p[\mu_{p-1}][\![1+p\mathbb{Z}_p]\!]$$

$$= \prod_{\varepsilon \in \hat{\mu}_{p-1}} \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] \simeq \prod_{\varepsilon \in \hat{\mu}_{p-1}} \mathbb{Z}_p[\![T]\!]$$

Thus, $W = \bigsqcup_{\varepsilon \in \hat{\mu}_{p-1}} D$ where $D =$ rigid space assoc to algebra

$$= \left\{ x \in \mathbb{C}_p : \forall f = \sum_{n\geq 0} a_n T^n \in \mathbb{Z}_p[\![T]\!] \text{ where } \sum_{n \geq 0} a_n x^n \text{ converges} \right\}$$

$$= \underline{m}_{\mathbb{C}_p} = \left\{ x \in \mathbb{C}_p \mid v(x) > 0 \right\}$$

Points of this wt space :

If $R = \mathbb{Q}_p$-algebra, p-adically complete

$$W(R) = \mathrm{Hom}_{\substack{\text{cont} \\ \mathbb{Z}_p\text{-alg}}}(\Lambda, R) = \mathrm{Hom}_{\substack{\text{cont} \\ \mathbb{Z}_p\text{-alg}}}(\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!], R)$$

$\shortparallel$

R-valued wts

$$= \mathrm{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, R^\times)$$

$\mathbb{Z} \hookrightarrow W(\mathbb{Q}_p)$

$k \mapsto (t \mapsto t^k)$, $t \in \mathbb{Z}_p^\times$

(2) Modular curves : fix $N \geq 5$, $p \nmid N$

$X_1(Np)_{/\mathbb{C}} = \Gamma_1(Np) \backslash \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$

$\downarrow$

$X(N,p)_{/\mathbb{C}} = \Gamma_1(N) \cap \Gamma_0(p) \backslash \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$

$\downarrow$

$X_1(N)_{/\mathbb{C}} = \Gamma_1(N) \backslash \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$

$\Gamma_1(Np) \subseteq \Gamma_1(N) \cap \Gamma_0(p) \subseteq \Gamma_1(N)$

$\Gamma_1(Np) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{ mod } Np \right\}$

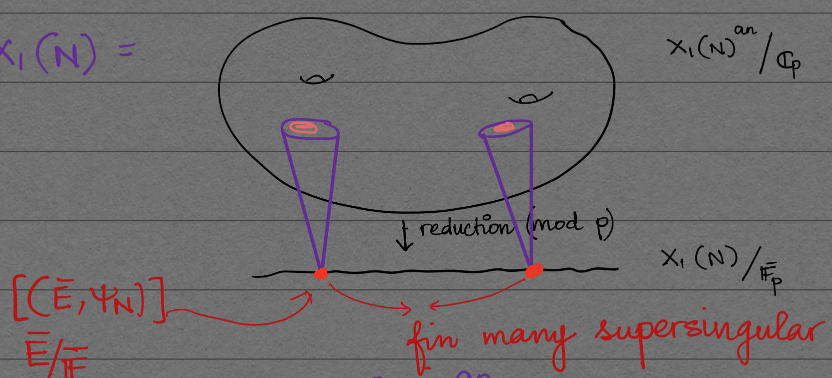
$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : p \mid c \right\}$

$$X_1(Np) = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}) \big/ \Gamma_1(Np) = \underbrace{\mathfrak{h} \big/ \Gamma_1(Np)}_{Y_1(Np)} \cup \mathbb{P}^1(\mathbb{Q}) \big/ \Gamma_1(Np)$$

$$Y_1(Np) \cup \{\text{cusps}\}$$

$$\begin{array}{lcl}
X_1(Np) & = & \{ [(E, \psi_N, \psi_p)] \mid \psi_p : \mu_p \hookrightarrow E[p] \} \\
\downarrow & & \\
X(N,p) & = & \{ [(E, \psi_N, C)] \mid C \subseteq E[p], \text{ subgroups of order } p \} \\
\downarrow & & \\
X_1(N)/\mathbb{C}_p & = & \{ [(E, \psi_N)] \mid E = \text{ell curve (generalized)}, \ \psi_N : \mu_N \hookrightarrow E[N] \} \quad \text{level } \Gamma_1(N)
\end{array}$$

$X_1(N) =$



Ques: What points in $X_1(N)^{an}/\mathbb{C}_p$ reduce to ss ell. curves

$$\begin{array}{l}
\mathcal{E} \quad \text{(generalised / universal ell curves)} \\
\downarrow \pi \longrightarrow \text{proper} \\
X_1(N)
\end{array}$$

modular sheaf

$$\underline{\omega}_{\mathcal{E}} := \pi_* \left(\Omega^1_{\mathcal{E}/X_1(N)} \right) \left(\log \left(\pi^{-1}(\text{cusps}) \right) \right)$$

$=$ line bundle on $X_1(N)$

$$k \in \mathbb{Z}, \quad M_k(\Gamma_1(N), R) = H^0\left(X_1(N)_R, \underline{\omega}^k_{\mathcal{E}, R}\right) = \begin{cases} \underline{\omega}_{\mathcal{E}}^{\otimes k} & k > 0 \\ \mathcal{O}_{X_1 N} & k = 0 \\ (\underline{\omega}^\vee_{\mathcal{E}})^{-k} & k < 0 \end{cases}$$

structure sheaf.

R = $\mathbb{Z}[1/N]$-algebra, $R = \mathbb{Q}_p$ or $\mathbb{C}_p$ or $\bar{\mathbb{F}}_p$ or ...

$E_{p-1} \in M_{p-1}(\Gamma_1(N), \mathbb{Z}_{(p)})$ [Hasse-Inv]

$E_{p-1}(x, \omega) = 0 \Leftrightarrow x = \text{supersingular pt}$

$\{x \in X_1(N)_{\mathbb{C}_p} \mid red(x) \in ss \text{ set}\} = \{x \in X_1(N)_{\mathbb{C}_p} \mid v(E_{p-1}(x,\omega)) > 0\}$

$X_1(N)^{ord} = \{x \in X_1(N)/\mathbb{C}_p \mid v(E_{p-1}(x,\omega)) = 0\}$

(Serre / Katz) • $\forall \lambda \in W(\mathbb{Q}_p)$, $\exists\ \underline{\omega}_{\mathcal{E}}^{ord,\lambda}$ = line bundle on X^{ord} with a property that if $\lambda = k \in \mathbb{Z}$, then

$\underline{\omega}_{\mathcal{E}}^{ord,k} = \underline{\omega}_{\mathcal{E}}^{k}\big|_{X^{ord}}$

• $M_\lambda^{p\text{-adic}}(\Gamma_1(N)) = H^0(X^{ord}, \underline{\omega}_{\mathcal{E}}^{ord,\lambda}) \xrightarrow{q\text{-exp}} \mathbb{Q}_p[[q]] \supseteq M^{p\text{-adic}}_{\lambda,\text{Serre}}$

*[cusps are in the ord locus]

$\lambda = k \in \mathbb{Z}$,

$\mathcal{M}_k^{classical}(\Gamma_1, N) = H^0(X_1(N), \underline{\omega}_{\mathcal{E}}^{k})$

$\circlearrowleft$ T_ℓ, T_p (Hecke) $\ell \nmid Np$

$\downarrow$ res

$H^0(X^{ord}, \underline{\omega}_{\mathcal{E}}^{k}) = \mathcal{M}_k^{p\text{-adic}}(\Gamma_1(N))$

$\circlearrowleft$ T_ℓ, U_p $\ell \nmid Np$

Remember: $M_\lambda^{p\text{-adic}}(\Gamma_1(N), \mathbb{Q}_p) = \mathbb{Q}_p$- Banach sp

There is no (known) compact Hecke operator

So, there is no fin slope decomp

Overconvergent modular forms (of integer wt)

$k \in \mathbb{Z} \subseteq W(\mathbb{Q}_p)$; $\mathcal{M}_k^{+}(\Gamma_1(N), \mathbb{Q}_p) = H^0(X_r, \underline{\omega}_{\mathcal{E}}^{k})$ → overconvergent mod forms of wt k, deg r

$\circlearrowleft$ T_ℓ, U_p (Hecke op) $\ell \nmid Np$

Let $r > 0$, $r \in \mathbb{Z}$;

$X_r := \{x \in X_1(N) \mid 0 \leq v(E_{p-1}(x,\omega)) < 1/r\}$

$X^{ord} \subsetneq X_r \subsetneq X_1(N)$

strict nbd of X^{ord}

$\mathcal{M}^{+}_{k,r}(\Gamma_1(N), \mathbb{Q}_p) = \mathbb{Q}_p$- Banach space & U_p is compact

If $h \geq 0$, a 'slope', we have a slope decompsn

$$\underbrace{\mathcal{M}^{+}_{k,r} \simeq \left(\mathcal{M}^{+}_{k,r}\right)^{(\leq h)}}_{\text{fin dim } \mathbb{Q}_p\text{-vs, preserved by } T_\ell} \oplus \left(\mathcal{M}^{+}_{k,r}\right)^{(>h)}$$

$\uparrow$ gen eigenvector for U_p

fin eigenvalues $s \in \mathbb{C}_p$, $v(s) \leq h$

fin dim $\mathbb{Q}_p$-vs, preserved by T_ℓ

* new material starts here [Coleman Theory]

$$\begin{array}{ccc} & [(E,\psi_N,H)] & \\ & \uparrow & X(N,p)^{an} \\ x & \swarrow & \downarrow p \\ [(E,\psi_N)] & & \\ r>0 \quad X_r & \subseteq & X_1^{an} \end{array}$$

Denote

$X(N,p)_r$ = Image of X_r

If $1/r < p/_{p+1}$, then $\forall x \in X_r$, $x = (E, \psi_N)$

$\exists$ a canonical subgroup $H \subseteq E[p]$ an order p subgp st. if E is ordinary

$H = E[p]^\circ$ connected

• $H (\text{mod } r) = \ker(\text{Frob})$

$$\mathcal{M}^{+}_{k,r}(\Gamma_1(N)) = H^0(X(N,p)_r, \underline{\omega}^k_{\mathcal{E}}) \xleftarrow{\text{res}} H^0(X(N,p), \underline{\omega}^k_{\mathcal{E}})$$

$k \in \mathbb{Z}$

$\hookrightarrow \mathcal{M}^{\text{cl}}_k(N,p)$

T_ℓ, $\ell \nmid Np$

U_p

Recall: λ = p-adic wt $(\lambda \in W(\mathbb{Q}_p))$

Suppose $\zeta_p^*(\lambda) \neq 0$, $\lambda \neq 1$

$$E^*_\lambda(q) = 1 + \frac{2}{\zeta_p^*(\lambda)} \sum_{n \geq 1} \sigma^*_{\lambda-1}(n) \quad , \quad \sigma^*_{\lambda-1}(n) = \sum_{\substack{d \mid n \\ (d,p)=1}} \lambda(d) d^{-1} \in \mathbb{Z}_p$$

$\uparrow$

$\mathcal{M}_\lambda^{p\text{-adic}}$

$\lambda = k \in \mathbb{Z}$, $E_k(q)$; E^*_k = p-stabilizer of E_k

$E_k^* \in \mathcal{M}_k^{cl}(N, p)$

$k \geq 4$

In gen, $E_k^* \in \mathcal{M}_{k,r}^{+}(N, p)$

$\forall\, \lambda \in W(\mathbb{Q}_p) \setminus \mathbb{Z}$, $E_\lambda^* \in \mathcal{M}_\lambda^{p\text{-adic}}(\Gamma_1(N))$

Thm[1] (Coleman) Let f be an overconvergent modular form of wt $k\,(\in \mathbb{Z}_{\geq 2})$ and slope $h < k-1$. Then, f is a classical modular form of wt k on $X(N, p)$

Thm[2] (Coleman) Let g be a classical eigenform of wt $k_0\,(\in \mathbb{Z}_{\geq 2})$. Denote by a_p the U_p-eigenvalue. Let $K/\mathbb{Q}_p$ be a fin extn containing all eigenvalues of g. If $a_p^2 \neq p^{k_0-1}$ and $v(a_p) < k_0 - 1$, then there is a closed disc $U \subseteq W$, $k_0 \in U$ and $F := \sum_{n \geq 0} A_n q^n \in A(U)[[q]]$ $\quad (A(U) = \mathcal{O}_W(U))$ st

(1) $\forall\, k \in U(K) \cap \mathbb{Z}$, $k > k_0$, $F(k) = \sum_{n=0}^{\infty} A_n(k) q^n \in K[[q]]$ is the q-expansion of a classical modular form, eigenform of level $\Gamma_1(N)$ and wt k (old)

(2) $F(k_0) = q$-expansion of g

We think of F as a p-adic eigen fam of forms deforming g

Suppose $\lambda \in U(K)$, $F_\lambda = \sum_{n=0}^{\infty} A_n(\lambda) q^n$ = p-adic modular form

Coleman proved:

F_λ are overconvergent modular forms of wt λ

$\lambda \in W(\mathbb{Q}_p) \setminus \mathbb{Z}$,

Suppose also that $\zeta_p^*(\lambda) \neq 0, \infty$

Let $f \in \mathcal{M}_\lambda^{p\text{-adic}}(N,p)$, $f \in H^0(X^{ord}, \underline{\omega}_{E}^{ord,\lambda})$

fix $r > 0$, $X^{ord} \subsetneq X_r$

We want to say f extends

defn: We say f is <u>overconvergent</u> if $\dfrac{f}{E_\lambda^*} \in H^0(X^{ord}, \underset{\mathcal{O}_{X_1(N)}}{\overset{\shortparallel}{\Omega^0}})$

extends to a section $H^0(X_r, \mathcal{O}_{X_1(N)})$

[This is the same as the previous one for $\lambda \in \mathbb{Z}$]

* There is extensive use of E_λ^* which does not generalize well

We then define modular sheaves $\underline{\omega}_E^\lambda$, $\lambda \in W(\mathbb{Q}_p) \setminus \mathbb{Z}$ on X_r, $r > 0$.

[We may do this later]