Tribonacci Numbers that are Products of Two Lucas Numbers

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Outline

- Preliminaries of Fibonacci and Lucas Numbers
- Work of Daşdemir-Emin: Fibonacci (resp. Lucas) numbers as a product of two Lucas (resp. Fibonacci) numbers.
- Work of Luca-Odjoumani-Togbé: Tribonacci Number as a product of two Fibonacci Numbers.
- New work: Tribonacci Numbers as products of two Lucas Numbers.

Fibonacci and Lucas Numbers

Definition: Fibonacci Numbers

The **Fibonacci numbers** are defined via the following recurrence relation

$$F_n = \begin{cases} 0 & \text{when } n = 0 \\ 1 & \text{when } n = 1 \\ F_{n-1} + F_{n-2} & \text{when } n \geq 2 \end{cases}$$

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The first ten Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Fibonacci and Lucas Numbers

Definition: Lucas Numbers

The **Lucas numbers** are defined as

$$L_n = \begin{cases} 2 & \text{when } n = 0 \\ 1 & \text{when } n = 1 \\ L_{n-1} + L_{n-2} & \text{when } n \geq 2. \end{cases}$$

Fibonacci and Lucas Numbers

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The first ten Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$$

Preliminaries

Characteristic Equation

Fibonacci and Lucas numbers are both second-order integer sequences satisfying

$$x^2 - x - 1 = 0.$$

This equation is called the **characteristic equation** of the Fibonacci (resp. Lucas) sequence.

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and

$$L_n = \alpha^n + \beta^n \quad \text{for the Lucas numbers.}$$

The Work of Daşdemir and Emin

Goal: To find all possible k, m, and n satisfying

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Theorem (Daşdemir-Emin, 2024)

Let k, m and n be positive integers. Then, the triples satisfying $F_k = L_m L_n$ are

k	m	n	F_k	L_m	L_n
1	1	1	1	1	1
2	1	1	1	1	1
4	1	2	3	1	3
8	2	4	21	3	7

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Today we will only see the sketch for $F_k = L_m L_n$. The other proof is similar.

Step 1: Upper bound for k in terms of n

- Using an induction argument we give upper and lower bounds of F_k in terms of α and β.

$$\alpha^{k-2} \leq F_k = L_m L_n \leq |\beta|^{-(n+m+2)}.$$

- Taking log on both sides

$$\begin{aligned}
(k-2)\log\alpha &\leq (-n-m-2)\log|\beta| \\
k &\leq 2 - \frac{(n+m+2)\log|\beta|}{\log\alpha} \\
&= 2 + (n+m+2) = 4 + n + m < 4n.
\end{aligned}$$

Step 2: Setting the stage to bound m

- Suppose that F_k can be written as a product of two Lucas numbers. Then

$$F_k = L_m L_n$$

$$\frac{\alpha^k - \beta^k}{\alpha - \beta} = (\alpha^m + \beta^m)(\alpha^n + \beta^n)$$

$$\vdots$$

$$\left|\sqrt{5}\alpha^{m+n-k} + \sqrt{5}\alpha^{m-k}\beta^n + \sqrt{5}\beta^m\alpha^{n-k} - \frac{\beta^k}{\alpha^k}\right| = |\alpha^{-k}\beta^{n+m}\sqrt{5} - 1|$$

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- Check that

$$0 < \Lambda_1 < \frac{8}{\alpha^{2m}}.$$

Step 3: Bounding m and Matveev's Theorem

- Applying a theorem by Matveev to Λ_1, we have

$$\begin{aligned}\log \Lambda_1 &> -1.4 \times 30^{3+3} \times 3^{4.5} \times 2^2 \times (1+\log 2) \times (1+\log 4n) \\ &\quad \times \log\alpha \times \log\alpha \times 2\log\sqrt{5} \\ &> -3.62 \times 10^{11} \times (1+\log 4n).\end{aligned}$$

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- Combining the two, we get

$$m < 3.77 \times 10^{11} (1+\log 4n).$$

Step 4: An upper bound for n

- We perform manipulations as before:

$$
\begin{aligned}
F_k &= L_m L_n \\
\frac{\alpha^k - \beta^k}{\alpha - \beta} &= L_m(\alpha^n + \beta^n) \\
&\vdots \\
\left|\sqrt{5}\alpha^{n-k}L_m - \frac{\beta^k}{\alpha^k}\right| &= \left|\alpha^{-k}|\beta|^n(\sqrt{5}L_m) - 1\right|
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- Repeating the same calculations as before, we have

$$0 < \Lambda_2 < \frac{33}{\alpha^n} \text{ and } n < 2.18 \times 10^{27}.$$

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In particular, define

$$\Gamma_1 := -k\log\alpha + (n+m)\log|\beta| + \log(\sqrt{5}).$$

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- Moreover

$$0 < \left|\frac{\Gamma_1}{\log|\beta|}\right| = \left|\frac{k\log\alpha}{\log|\beta|} - (n+m) + \frac{\log\left(\frac{1}{\sqrt{5}}\right)}{\log|\beta|}\right| < \frac{34}{\alpha^{2m}}.$$

Lemma of Dujella-Pethö

Technical Lemma (1998)

Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be positive rational numbers with $A > 0$ and $B > 1$. Let $\epsilon = \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\epsilon > 0$, then there is **no integer solution** (x, y, z) of inequality

$$0 < x\tau - y + \mu < AB^{-z} \text{ where } x \leq M \text{ and } z \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

How we apply the lemma

- We apply the lemma to

$$0 < \left| k\left(\frac{\log \alpha}{\log |\beta|}\right) - (n+m) + \frac{\log\left(\frac{1}{\sqrt{5}}\right)}{\log |\beta|} \right| < 34(\alpha^2)^{-m}.$$

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- By comparison: $A = 34$, $B = \alpha^2$, $z = m$, and $\mu_m = \frac{(1/\sqrt{5})}{\log |\beta|} > 0$.
- Set $M = 9.1 \times 10^{27}$ (chosen so that $k < 4n < M$)

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- Set $M = 9.1 \times 10^{27}$ (chosen so that $k < 4n < M$) and $\tau = \frac{\log\alpha}{\log|\beta|}$. The continued fraction expansions of τ yields

$$\frac{p_{47}}{q_{47}} = \frac{1394991136110806534618331 1454}{921342236120432337936155 16979}.$$

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- Check that $6M < q_{47}$ and define

$$\epsilon = ||\mu_m q_{47}|| - M||\tau q_{47}|| > 0.486.$$

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$$0 < \left| k \left(\frac{\log \alpha}{\log |\beta|} \right) - (n+m) + \frac{\log \left(\frac{1}{\sqrt{5}} \right)}{\log |\beta|} \right| < 34(\alpha^2)^{-m}.$$

- By comparison: $A = 34$, $B = \alpha^2$, $z = m$, and $\mu_m = \frac{(1/\sqrt{5})}{\log |\beta|} > 0$.
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$$\frac{p_{47}}{q_{47}} = \frac{139499113611080653461833311454}{92134223612043233793615516979}.$$

- Check that $6M < q_{47}$ and define

$$\epsilon = ||\mu_m q_{47}|| - M||\tau q_{47}|| > 0.486.$$

- The lemma of Dujella-Pethö forces that $m \leq 73$.

Better bounds for n and completing the proof

- A similar manipulation can now be repeated with Λ_2 instead of Λ_1. Once again, using the lemma of Dujella-Pethö, the bounds on n can be improved to $n \leq 160$.

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- At this point, it is a (small) finite check, which can be done on (say) Mathematica over the range $m \leq 75$ and $n \leq 160$ to determine all possible solutions for $F_k = L_m L_n$.

The work of Luca, Odjoumani and Togbé

Definition: Tribonacci Numbers

The **Tribonacci numbers** are defined via the following recurrence relation

$$T_n = \begin{cases} 0 & \text{when } n = 0 \\ 1 & \text{when } n = 1, 2 \\ T_{n-1} + T_{n-2} + T_{n-3} & \text{when } n \geq 3. \end{cases}$$

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The first ten Tribonacci numbers are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots$$

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Characteristic equation

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Its characteristic equation is $X^3 - X^2 - X - 1$ with roots

$$\gamma = \frac{1 + r_1 + r_2}{3}, \quad \delta, \bar{\delta} = \frac{2 - (r_1 + r_2) \pm i\sqrt{3}(r_1 - r_2)}{6},$$

where

$$r_1 = \sqrt[3]{19 + 3\sqrt{33}} \quad \text{and} \quad r_2 = \sqrt[3]{19 - 3\sqrt{33}}.$$

Preliminaries

Binet's Formula

For the Tribonacci numbers, the Binet's formula is

$$T_n = a\gamma^n + b\delta^n + \bar{b}\bar{\delta}^n$$

where

$$a = \frac{5\gamma^2 - 3\gamma - 4}{22} \quad \text{and} \quad b = \frac{5\delta^2 - 3\delta - 4}{22}.$$

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The idea of the proof is similar to the previous work but we discuss some of the differences as the new work (to be discussed at the end) uses some of these ideas.

The work of Luca, Odjoumani, and Togbé

Theorem (Luca-Odjoumani-Togbé, 2024 (Fibonacci Quarterly))

For positive integers k, m and n, the triples satisfying $T_k = F_m F_n$ are

k	m	n	T_k	F_m	F_n
1	1	1	1	1	1
1	2	2	1	1	1
1	1	2	1	1	1
2	1	1	1	1	1
2	2	2	1	1	1
2	1	2	1	1	1
3	1	3	2	1	2
3	2	3	2	1	2
4	3	3	4	2	2
6	1	7	13	1	13
6	2	7	13	1	13
7	4	6	24	3	8

My contribution: work in progress

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Theorem (Q.)

Let k, m and n be positive integers. Then, the triples satisfying $T_k = L_m L_n$ are

k	m	n	T_k	L_m	L_n
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4	1	3	4	1	4
5	1	4	7	1	7
8	3	5	44	4	11

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The idea of the proof parallels the previous results.

Step 1: Upper bound for k in terms of n

- Using an induction argument we give upper and lower bounds of T_k in terms of γ and β.

$$\gamma^{k-2} < T_k = L_m L_n < |\beta|^{-(n+m+2)} \qquad \text{and} \qquad |\beta|^{-(m+n-2)} < \gamma^{k-1}$$

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- Taking logs on both sides yield upper and lower bounds of k

$$\frac{\log|\beta|}{\log\gamma}(-m-n) + 0.2 < k < \frac{\log|\beta|}{\log\gamma}(-m-n) + 3.8.$$

Step 2: Setting the stage to bound m

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$$\begin{aligned} T_k &= L_m L_n \\ a\gamma^k + b\delta^k + \bar{b}\bar{\delta}^k &= (\alpha^m + \beta^m)(\alpha^n + \beta^n) \\ &\vdots \end{aligned}$$

- Let's define

$$\Lambda_1 := |a\gamma^k \beta^{-(m+n)} - 1|$$

$$\begin{aligned} |a\gamma^k \beta^{-(m+n)} - 1| &= \big| -(b\delta^k + \bar{b}\bar{\delta}^k)\beta^{-(m+n)} + \alpha^{m+n}\beta^{-(m+n)} + \alpha^m\beta^n\beta^{-(m+n)} \\ &\quad + \alpha^n\beta^m\beta^{m+n}\big| \\ &\neq 0 \end{aligned}$$

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Now, taking log on both sides

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$$\Lambda_1 := |a\gamma^k|\beta|^{-(m+n)} - 1| < \frac{5.62}{|\beta|^{2m}}.$$

Now, taking log on both sides

$$\log \Lambda_1 < \log 5.62 - 2m \log |\beta|.$$

- Combining the two, we get

$$m \log |\beta| < 3.85 \times 10^{14} (\log m + n).$$

Step 4: An upper bound for n

- We perform manipulations as before

$$T_k = L_m L_n$$

$$a\gamma^k + b\delta^k + \bar{b}\bar{\delta}^k = L_m(\alpha^n + \beta^n)$$

$$\vdots$$

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- Define

$$
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- I have checked till $k = 5000$ and that gives me good reason to believe that the table present in my result is in fact complete.

Thank You!
Any Questions? :)

Matveev's Theorem

Theorem (Matveev)

The following inequality holds for any non-zero Λ *in the real field* $\mathbb{F}$

$$\log|\Lambda| > -1.4\times 30^{s+3}\times s^{4.5}\times D^2\times(1+\log D)\times(1+\log B)\times A_1\times A_2\times\cdots\times A_s.$$

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- **NB**: Every irrational number can be expressed as in infinite continued fraction known as convergents. The n-th convergent is given by

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$$c_n = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-1} + q_{n-2}}$$

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where $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, $q_0 = 1$

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