# Local Cohomology

§ Motivation (Introduction (Boxer–Pilloni))

§ Topological Background (§2 of B–P).

## § 1.1 Siegel Shimura Varieties

$G =$ reductive gp $/\mathbb{Q}$ $\longrightarrow$ $(G, X)$ = Shimura datum

complex analytic sp with $G(\mathbb{R})$

Ex: $G = GL_2/\mathbb{Q}$, then $X = \mathfrak{h}^+ \sqcup \mathfrak{h}^-$

$G = GSp_{2n}/\mathbb{Q}$, then $X = \mathfrak{h}_n^+ \sqcup \mathfrak{h}_n^-$

Siegel space

Let $J = \begin{pmatrix} & I \\ -I & \end{pmatrix} \in GL_{2n} \longrightarrow GSp_{2n} = \{ g \in GL_{2n} : {}^t g J g = \mu J, \exists \mu \in \mathbb{G}_m \}$

$\mathfrak{h}_n^+ = \{ M \in M_{n \times n}(\mathbb{C}) : {}^t M = M \text{ \& } \mathrm{Im}(M) \gg 0 \}$

Associated to Shimura datum we have a Shimura var $S_K$

$K \subseteq G(\mathbb{A}_f)$ sufficiently nice, $S_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$ = complex manifold

For us: $G = GSp_{2n}/\mathbb{Q}$, $S_K(\mathbb{C}) = \bigsqcup_{\Gamma} \Gamma \backslash \mathfrak{h}_n^+$ $\quad \Gamma \leq G(\mathbb{Q})$

$S_K$ is a scheme $/\mathbb{Q}$

moduli space of Ab var with extra structure.

Consider $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}}$

$z = a + ib \mapsto \begin{pmatrix} aI & bI \\ -bI & aI \end{pmatrix}$

$X \longleftrightarrow G(\mathbb{R})$ – conjugacy class of $h$

## §1.2 Flag Variety

$$\mu : \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{h} G(\mathbb{C})$$
$$z \mapsto (z, 1) \longmapsto \begin{pmatrix} zI & 0 \\ 0 & I \end{pmatrix}$$

Assoc to $\mu$, there is a parabolic $P_\mu$

$\left[ G/P_\mu = \text{projective var} \right]$
$=: \mathcal{Fl}_\mu / \mathbb{Q}$

$K|\mathbb{Q}$ = extn

$$\mathcal{Fl}_\mu(K) := \left\{ \underset{\substack{\text{maximal} \\ \text{isotropic subsp} \\ \text{ie } \dim U = n}}{U} \subseteq V_{std} \simeq K^{2n} \right\}$$

$P_\mu$ = Siegel parabolic $= M_\mu U_\mu$

$M_\mu$: Levi $= GL_n \times GL_1$ $\quad \left( \begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array} \right)$

$X \hookrightarrow \mathcal{Fl}_\mu(\mathbb{C})$ (relevant for the workshop)

Borel embedding

## §1.3 Automorphic Vector Bundles

In $GL_2 \rightsquigarrow H^0(X, \omega^{\otimes k})$

What happens for $GSp_{2n}$?

**Prop:** We have a functor $\qquad S_K^{tor} \leftarrow$ toroidal compactification (non-canon)

$$Rep_{\mathbb{Q}}(M_\mu) \to VB(S_K)$$
$$V \longmapsto \mathcal{V}_K$$

$Rep_{\mathbb{Q}}(M_\mu)$: algebraic repns (fin dim)

$VB(S_K)$: vector bundle : loc free sheaf of fin rk $/ S_K$

Rk. $S_K \xleftarrow{\pi} A_K$ (Universal Abelian scheme)

Start with standard repn of $GL_n$, say $V$

$$V \longmapsto \underline{\omega} = \pi_* \Omega^1_{A_K/S_K} \qquad \underline{\omega} = \backslash\text{omega}$$

eg: $n = 2$, $V^\lambda = Sym^\lambda(Std) \mapsto \mathcal{V}^\lambda = Sym^\lambda \underline{\omega}$

$\lambda \in \mathbb{Z}$

(Boxer–Pilloni) $K^p \subseteq G(\mathbb{A}_f^p)$

fixed, 'nice enough'

Colim $K_p \subseteq G(\mathbb{Q}_p)$ $\quad H^i(S^{tor}_{K^pK_p}, \mathcal{V}^\lambda) =: H^i(K_p, \lambda)$

$G_{\mathbb{Q}_p}$-repn $\qquad \qquad H^i(K_p,\lambda) \supseteq H^i(K_p,\lambda)^{fs}$ ← fin slope

These $H^i(K_p,\lambda)^{fs}$ is calculated from $H^i_{Z_d/Z_{d+1}}(S_K^{tor,ad}, \lambda)^{fs}$

$Z_d$ = nbd of pre-images of $\pi_{HT}$

$S^{tor,ad}_{K^p}/\mathbb{C}_p$ = adic-space $/\mathbb{C}_p$

$\pi_{HT} : S^{tor,ad}_{K^p} \to \mathcal{F}l^{ad}/_{\mathbb{C}_p}$

Drinfeld upper half plane

Idea: for $Gl_2$, $\mathcal{F}l \simeq \mathbb{P}^1 = \mathbb{P}^1(\mathbb{Q}_p) \sqcup \mathfrak{h}_D$

$\rightarrow \pi_{HT}^{-1}(\{u\})$, $u \in \mathbb{P}^1(\mathbb{Q}_p)$ $\qquad = \bigsqcup_{w} B \cap BwB \,/\, B = \{Id\} \sqcup \_$

Rk: Coleman Theory studies $H^0(K_p,\lambda)^{ps}$ as $\lambda$ varies p-adically

$\mathcal{W} \longleftrightarrow \Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ (from Adrian's lecture)

p-adic space

But now,

$\mathcal{W} \leftrightarrow \Lambda = \mathbb{Z}_p[[T_\mu(\mathbb{Z}_p)]] \simeq \mathbb{Z}_p[[(\mathbb{Z}_p^\times)^{n+1}]]$

$\mathrm{Spa}(\Lambda,\Lambda) \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ $\qquad \mu(\mathbb{G}_m) \subseteq T_\mu \subseteq M_\mu$

$\mathcal{W}$ has the property that

$$(A,A^+) \longrightarrow \mathrm{Hom}(\mathrm{Spa}(A,A^+), \mathcal{W}) = \mathrm{Hom}_{cont}(T_\mu(\mathbb{Z}_p), A)$$

## §2 Topological Background: Let $X$ = top space

Associate $Ab_X$ = Abelian Sheaves

(One can consider derived category $\mathcal{D}(Ab_X) = \mathcal{C}(Ab_X)$)

complexes

Let $i : Z \to X$ closed subspace

Define: 1. $\Gamma_Z(X, \mathcal{F}) = \{ s \in \Gamma(X, \mathcal{F}) : \mathrm{Supp}(s) \subseteq Z \}$

2. If $Z$ = loc. closed

$\Gamma_Z(X, \mathcal{F}) := \Gamma_Z(V, \mathcal{F}|_V) \qquad Z \subseteq_c V \subseteq_o X$

3. $R\Gamma_Z(X, -) : \mathcal{D}(Ab_X) \to Ab_X$

Rk. Suppose that $i : Z \to X$ is closed. Then $i_* : Ab_Z \to Ab_X$ given by $\mathcal{F} \mapsto (U \mapsto \mathcal{F}|_{i^{-1}U})$

This has a right adjoint

$$i^! : Ab_X \to Ab_Z$$

$$\mathcal{F} \mapsto (W \mapsto \Gamma_W(W', \mathcal{F}|_W))$$

where $W' \subseteq_o X$, $W' \cap Z = W$

Prop: $R\Gamma_Z(X, \mathcal{F}) = R\Gamma(X, i_* R i^! \mathcal{F})$

Properties: (1) $\Gamma_Z(X, -)$ is functorial on $\mathcal{F}$

(2) $Z \subseteq Z' \subseteq X$ closed (not nec) then

$$0 \to \Gamma_Z(X, \mathcal{F}) \to \Gamma_{Z'}(X, \mathcal{F}) \to \Gamma_{Z'-Z}(X, \mathcal{F}) \to 0 \text{ is exact}$$

This yields

$$R\Gamma_Z(X, \mathcal{F}) \to R\Gamma_{Z'}(X, \mathcal{F})$$

(3) fix $U = X - Z$, we have an exact triangle

$$R\Gamma_Z(X, \mathcal{F}) \to R\Gamma(X, \mathcal{F}) \to R\Gamma(U, \mathcal{F}) \xrightarrow{+}$$

Assoc to the $\Delta$, we have an exact seq$^n$ in cohomology

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1_Z(X, \mathcal{F}) \to \cdots$$

(4) $Z_1, Z_2$ disjoint

$$R_{Z_1}(X,\mathcal{F}) \oplus R_{Z_2}(X,\mathcal{F})$$
$$\downarrow \text{quasi-iso}$$
$$R_{Z_1 \cup Z_2}(X,\mathcal{F})$$

## §2.2 Important Spectral Seq$^n$

$W \subseteq Z \subsetneq X$, $\quad R\Gamma_{Z/W}(X,\mathcal{F}) := R\Gamma_{Z-W}(X-W,\mathcal{F})$

If $Z \subseteq Z'$ and $W \subseteq W'$, then

$$R\Gamma_{Z-W}(X,\mathcal{F}) \to R\Gamma_{Z'-W'}(X,\mathcal{F})$$

Prop: (1) If $Z_3 \subseteq Z_2 \subseteq Z_1$, then we have an exact $\Delta$

$$R_{Z_2-Z_3}(X,\mathcal{F}) \to R_{Z_1-Z_3}(X,\mathcal{F}) \to R_{Z_1-Z_2}(X,\mathcal{F})$$

(2) If $X = Z_0 \supseteq Z_1 \supseteq \dots \supseteq Z_r = \phi$

then there is a degenerate spectral seq$^n$

$$E_1^{p,q} = H^{p+q}_{Z_p/Z_{p+1}}(X,\mathcal{F}) \Rightarrow H^{p+q}(X,\mathcal{F})$$

Rk: This works well if one works with $(X,\mathcal{O}_X)$ and the category of coherent sheafs.

(Ex: If $Z \subseteq X$ closed subscheme

$\Gamma_Z^{coh}(-,\mathcal{F}) := \ker(\mathcal{F} \to \mathcal{H}om(\mathcal{I},\mathcal{F}))$

Then, the global sections $\Gamma_Z^{coh}(X,\mathcal{F}) = \Gamma_Z(X,\mathcal{F})$)