

# GROWTH OF $p$ -PARTS OF IDEAL CLASS GROUPS AND FINE SELMER GROUPS IN $\mathbb{Z}_q$ -EXTENSIONS WITH $p \neq q$

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ABSTRACT. Fix two distinct odd primes  $p$  and  $q$ . We study " $p \neq q$ " Iwasawa theory in two different settings.

- (1) Let  $K$  be an imaginary quadratic field of class number 1 such that both  $p$  and  $q$  split in  $K$ . We show that under appropriate hypotheses, the  $p$ -part of the ideal class groups is bounded over finite subextensions of an anticyclotomic  $\mathbb{Z}_q$ -extension of  $K$ .
- (2) Let  $F$  be a number field and let  $A/F$  be an abelian variety with  $A[p] \subseteq A(F)$ . We give sufficient conditions for the  $p$ -part of the fine Selmer groups of  $A$  over finite subextensions of a  $\mathbb{Z}_q$ -extension of  $F$  to stabilize.

## 1. INTRODUCTION

Let  $F/\mathbb{Q}$  be an algebraic number field and  $F_\infty/F$  be a Galois extension with Galois group isomorphic to the additive group  $\mathbb{Z}_q$  of  $q$ -adic integers. For each integer  $n \geq 0$ , there is a unique subfield  $F_n/F$  of degree  $q^n$ . Let  $h(F_n)$  be the class number of  $F_n$ . K. Iwasawa showed that if  $q^{e_n}$  is the highest power of  $q$  dividing  $h(F_n)$ , then there exist integers  $\lambda, \mu, \nu$  independent of  $n$ , such that  $e_n = \mu q^n + \lambda n + \nu$  for  $n \gg 0$ . On the other hand, in [Was75, Was78], L. C. Washington proved that for distinct primes  $p$  and  $q$ , the  $p$ -part of the class number stabilizes in the *cyclotomic*  $\mathbb{Z}_q$ -extension of an abelian number field. Washington's results have been extended to other  $\mathbb{Z}_q$ -extensions where primes are finitely decomposed. In particular, J. Lamplugh proved the following in [Lam15]: if  $p, q$  are distinct primes  $\geq 5$  that split in an imaginary quadratic field  $K$  of class number 1 and  $F/K$  is a prime-to- $p$  abelian extension which is also unramified at  $p$ , then the  $p$ -class group stabilizes in the  $\mathbb{Z}_q$ -extension of  $F$  which is unramified outside precisely one of the primes above  $q$ . There have also been speculations by J. Coates on the size of the whole class group in a cyclotomic tower; see [Coa12], especially the discussion in §3 and Conjecture D.

Let  $p$  and  $q$  be two distinct odd primes and  $K$  an imaginary quadratic field of class number 1 in which both  $p$  and  $q$  split. We write  $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$  and  $q\mathcal{O}_K = \mathfrak{q}\bar{\mathfrak{q}}$ . Given an ideal  $\mathfrak{h}$  of  $\mathcal{O}_K$ , we write  $\mathcal{R}(\mathfrak{h})$  for the ray class field of  $K$  of conductor  $\mathfrak{h}$ . In the first half of this article, we study the growth of the  $p$ -part of the ideal class group in a  $\mathbb{Z}_q$ -anticyclotomic tower. This generalizes [Lam15, Theorem 1.3], where the stability of the  $p$ -part of the class numbers  $\mathcal{R}(\mathfrak{g}\mathfrak{q}^n)$  is studied. More precisely, we prove the following result.

**Theorem A.** *Let  $K$  be an imaginary quadratic field of class number 1. Let  $p$  and  $q$  be distinct primes ( $\geq 5$ ) which split in  $K$ . Let  $\mathfrak{r}$  be a fixed ideal of  $\mathcal{O}_K$  coprime to  $pq$  such that  $\mathfrak{r}$  is a product of split primes<sup>1</sup>. Let  $\mathcal{F} = \mathcal{R}(\mathfrak{r}\mathfrak{q})$ . We assume that  $p \nmid [\mathcal{F} : K]$ . Let  $\mathcal{R}(\mathfrak{r}\mathfrak{q}^\infty)^{\text{ac}}/\mathcal{F}$  denote the anticyclotomic  $\mathbb{Z}_q$ -extension and write  $\mathcal{F}_n$  for the unique subextension of  $\mathcal{R}(\mathfrak{r}\mathfrak{q}^\infty)^{\text{ac}}/\mathcal{F}$  whose degree is  $q^n$ . Then there exists an integer  $N$  such that for all  $n \geq N$ ,*

$$\text{ord}_p(h(\mathcal{F}_n)) = \text{ord}_p(h(\mathcal{F}_N)).$$

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<sup>1</sup>In this article, a split prime of  $K$  refers to a prime ideal of  $\mathcal{O}_K$  that lies above a rational prime that splits in  $K$ .

The hypothesis on  $\mathfrak{r}$  being a product of split primes is crucial for the use of a theorem of H. Hida, which guarantees the non-vanishing modulo  $p$  of the algebraic  $L$ -values of anticyclotomic characters factoring through  $\mathcal{R}(\mathfrak{r}q^\infty)^{\text{ac}}$  (see Theorem 3.2). To prove Theorem A, we link this non-vanishing to the stabilization of the  $p$ -class groups via the ( $p$ -adic) Iwasawa main conjecture proved by K. Rubin [Rub91]. Our strategy is inspired by the work of Lamplugh [Lam15], which we outline below.

In §2, we introduce an auxiliary elliptic curve  $E/K$  with CM by  $\mathcal{O}_K$  such that the conductor  $\mathfrak{f}$  of its Hecke character is a product of split primes in  $K$  with  $p \nmid [\mathcal{R}(\mathfrak{f}) : K]$ . Let  $\mathfrak{g} = \text{lcm}(\mathfrak{f}, \mathfrak{r})$ . By a result of Lamplugh, when the algebraic  $L$ -value of certain Hecke character is nonzero modulo  $p$ , the corresponding modules of local  $p$ -adic units and elliptic units over an extension generated by  $E[\mathfrak{p}^\infty]$  coincide after taking appropriate isotypic components (see Theorem 4.2 for the precise statement). Combining this with Hida's theorem, we prove in Theorem 4.3 that the  $p$ -primary Galois modules featured in the Iwasawa main conjecture stabilize in the anticyclotomic  $\mathbb{Z}_q$ -extension  $\mathcal{R}(\mathfrak{g}q^\infty)^{\text{ac}}/\mathcal{R}(\mathfrak{g}q)$ . This can be translated into a statement on  $p$ -class groups, proving a special case of Theorem A, where the ideal  $\mathfrak{r}$  is divisible by  $\mathfrak{f}$  (see Theorem 4.4). To complete the proof of Theorem A, we bound the  $p$ -class groups over the tower  $\mathcal{R}(\mathfrak{r}q^\infty)^{\text{ac}}/\mathcal{R}(\mathfrak{r}q)$  by those over  $\mathcal{R}(\mathfrak{g}q^\infty)^{\text{ac}}/\mathcal{R}(\mathfrak{g}q)$ .

In the second half of the article, we prove a general statement (see Theorem 5.3) which shows that in certain  $\mathbb{Z}_q$ -extensions of a number field  $F$ , the growth of the  $p$ -part of the class group is closely related to that of the  $p$ -primary *fine* Selmer group of an abelian variety  $A/F$ . This subgroup of the classical  $p$ -primary Selmer group is denoted by  $\text{Sel}_0(A/F)$ , and is obtained by imposing stronger vanishing conditions at primes above  $p$  (the precise definition is reviewed in §5.1). The following result is an application of the aforementioned theorem to the growth of the  $p$ -part of fine Selmer group of a fixed abelian variety  $A$  over a  $\mathbb{Z}_q$ -tower (which is not necessarily anticyclotomic).

**Theorem B.** *Let  $p$  and  $q$  be distinct odd primes. Let  $F$  be any number field and  $A/F$  be an abelian variety such that  $A[p] \subseteq A(F)$ . Let  $F_\infty/F$  be a  $\mathbb{Z}_q$ -extension where the primes above  $q$  and the primes of bad reduction of  $A$  are finitely decomposed. If there exists  $N \geq 0$  such that for all  $n \geq N$ ,*

$$\text{ord}_p(h(F_n)) = \text{ord}_p(h(F_N)),$$

*then there exists an integer  $N' \geq N$  such that for all  $n \geq N'$ , there is an isomorphism*

$$\text{Sel}_0(A/F_n) \simeq \text{Sel}_0(A/F_{N'}).$$

In particular, Theorem B applies to the setting studied by Washington [Was75, Was78]. Finally, we remark that unlike what we have found for fine Selmer groups in Theorem B, it has been shown by T. Dokchitser and V. Dokchitser that the  $p$ -part of the Tate–Shafarevich group of an abelian variety in a  $\mathbb{Z}_q$ -tower can be unbounded; see [DD15, Example 1.5].

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## 2. FINDING AUXILIARY CM ELLIPTIC CURVES

Let  $K = \mathbb{Q}(\sqrt{-d})$  be an imaginary quadratic field of class number 1. As discussed in the introduction, we shall work with an auxiliary CM elliptic curve  $E/K$  in order to prove Theorem A. Recall that the imaginary quadratic fields of class number 1 are precisely the following

$$\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11}), \mathbb{Q}(\sqrt{-19}), \mathbb{Q}(\sqrt{-43}), \mathbb{Q}(\sqrt{-67}), \mathbb{Q}(\sqrt{-163}).$$

For each choice of  $K$ , we shall write down an explicit elliptic curve  $E/K$  such that

- (a)  $E$  has CM by  $\mathcal{O}_K$ ;
- (b) If  $\mathfrak{f}$  denotes the conductor of the Hecke character  $\psi$  attached to  $E$ , then  $\mathfrak{f}$  is only divisible by split primes of  $K$ ;

(c) The rational primes dividing  $[\mathcal{R}(f) : K]$  are either 2, 3 or primes that are non-split in  $K$ .

We remark that condition (c) ensures that the prime  $p$  in the statement of Theorem A does not divide  $[\mathcal{R}(f) : K]$ .

If  $E/K$  is an elliptic curve with CM by an order  $\mathcal{O}$  in  $K$ , the  $j$ -invariant  $j(E)$  is an integer in this case, so  $E$  must be a twist of the base extension of an elliptic curve  $A/\mathbb{Q}$ . For  $d > 3$ ,  $A$  is uniquely determined (up to isomorphism over  $K$ ) by the condition that it has CM by  $\mathcal{O}_K$  and its base change to  $K$  has bad reduction at the ramified prime  $\mathfrak{P} = (\sqrt{-d})$ . For  $d = 1, 2$ , and 3 there are several choices for the elliptic curve over  $\mathbb{Q}$  (see [CP19, Remark 3.1]).

When  $d > 3$ , it follows from [CP19, Theorem 3.3] that if we twist  $A/K$  by a character corresponding to  $K(\sqrt{\alpha})$  where  $\alpha = \mathfrak{P}\mathfrak{Q}$  such that  $\mathfrak{Q}$  is a prime of  $K$  distinct from  $\mathfrak{P}$  satisfying  $\mathfrak{Q} \equiv u^2\sqrt{-d} \pmod{4\mathcal{O}_K}$  for some  $u \in \mathcal{O}_K$ , then the twisted elliptic curve (over  $K$ ) has good reduction everywhere except at  $\mathfrak{Q}$ . Therefore, for our purposes, it is enough to find such  $\mathfrak{Q}$  that is a split prime in  $K$ . Indeed, we may choose  $r \in \mathbb{Z}$  such that  $(4r + \sqrt{-d})(4r - \sqrt{-d}) = 16r^2 + d$  is an odd rational prime. Such  $r$  exists for all possible values of  $d$ . For example, we may take  $r$  to be 1 when  $d = 43, 67, 163$ . Then  $4r + \sqrt{-d}$  is a split prime of  $K$  and  $4r + \sqrt{-d} \equiv 1^2\sqrt{-d} \pmod{4\mathcal{O}_K}$ . In particular, we may apply [CP19, Theorem 3.3] with  $\mathfrak{Q} = (4r + \sqrt{-d})$  and  $u = 1$ , resulting in a CM curve  $E$  satisfying properties (a) and (b) above.

When  $d < 43$ , we find  $E/K$  by inspection using the data available on [LMF23]. In all our examples below,  $E/K$  has bad reduction at one or two split primes which are coprime to 6. In particular, the conductor of  $E/K$  is given by square of the product of the bad prime(s), whereas the conductor  $\mathfrak{f}$  of the Hecke character  $\psi$  attached to  $E$  is given by the product of the bad prime(s) (see [ST68, Theorem 12]). The ray class group  $\text{Gal}(\mathcal{R}(f)/K)$  (and hence  $[\mathcal{R}(f) : K]$ ) is computed using MAGMA [BCP97].

$d$	$A/\mathbb{Q}$	base change curve $A/K$	twisted curve $E/K$	bad prime(s) of $E/K$	$[\mathcal{R}(f) : K]$
1	64.a4	2.0.4.1-256.1-CMa1	2.0.4.1-25.1-CMa1	$2 + \sqrt{-1}$	1
2	256.d1	2.0.8.1-1024.1-CMb1	2.0.8.1-9.3-CMa1	$1 - \sqrt{-2}$	1
3	27.a4	2.0.3.1-81.1-CMa1	2.0.3.1-2401.3-CMa1	$2 \pm \sqrt{-3}$	6
7	49.a4	2.0.7.1-49.1-CMa1	2.0.7.1-1849.1-CMa1	$6 - \sqrt{-7}$	21
11	121.b2	2.0.11.1-121.1-CMa1	2.0.11.1-9.1-CMa1	$\frac{-1-\sqrt{-11}}{2}$	1
19	361.a2	2.0.19.1-361.1-CMa1	2.0.19.1-49.3-CMa1	$\frac{-1+\sqrt{-19}}{2}$	3
43	1849.b2	2.0.43.1-1849.1-CMa1	$\alpha = -43 + 4\sqrt{-43}$	$4 + \sqrt{-43}$	29
67	4489.b2	2.0.67.1-4489.1-CMa1	$\alpha = -67 + 4\sqrt{-67}$	$4 + \sqrt{-67}$	41
163	26569.a2	curve not in database	$\alpha = -163 + 4\sqrt{-163}$	$4 + \sqrt{-163}$	89

### 3. A RESULT OF HIDA ON $L$ -VALUES OF ANTICYCLOTOMIC HECKE CHARACTERS

Throughout this section and the next,  $K$  is a fixed imaginary quadratic field of class number 1. We fix an elliptic curve  $E/K$  with CM by  $\mathcal{O}_K$  as given in §2. Recall that  $\psi$  denotes the Hecke character over  $K$  with conductor  $\mathfrak{f}$  attached to  $E$ .

We review a special case of a result of Hida from [Hid07] that will play a crucial role in our proof of Theorem A.

**Definition 3.1.** Let  $\mathfrak{h}$  be any integral ideal of  $K$  and let  $\epsilon$  be any Hecke character of  $K$ . The  $\mathfrak{h}$ -*imprimitive*  $L$ -function of  $\epsilon$  is defined as follows

$$\begin{aligned}
 L_{\mathfrak{h}}(\epsilon, s) &= \prod_{\text{gcd}(\nu, \mathfrak{h})=1} \left( 1 - \frac{\epsilon(\nu)}{(N\nu)^s} \right)^{-1} \\
 &= \sum_{\text{gcd}(\mathfrak{a}, \mathfrak{h})=1} \frac{\epsilon(\mathfrak{a})}{(N\mathfrak{a})^s},
 \end{aligned}$$

where the product runs over *prime ideals*  $\nu$  of  $K$  coprime to  $\mathfrak{h}$ , and sum is taken over *integral ideals*  $\mathfrak{a}$  coprime to  $\mathfrak{h}$ .

Fix an integral ideal  $\mathfrak{g}$  of  $K$  which is divisible by  $\mathfrak{f}$ , relatively prime to  $pq$ , and such that only split primes of  $K$  divide  $\mathfrak{g}$ . Let  $F = \mathcal{R}(\mathfrak{g}q)$  be the *ray class field* of  $K$  of conductor  $\mathfrak{g}q$  and write  $\Delta = \text{Gal}(F/K)$ . Set  $F_\infty = \bigcup_{n \geq 1} \mathcal{R}(\mathfrak{g}q^n)$ ; this is a  $\mathbb{Z}_q^2$ -extension of  $F$ . We fix an isomorphism

$$\text{Gal}(F_\infty/K) \simeq \text{Gal}(F/K) \times \text{Gal}(K_\infty/K) = \Delta \times \mathbb{Z}_q^2.$$

Let  $\epsilon$  be a character of  $\text{Gal}(F_\infty/K)$ . For our purpose,  $\epsilon$  will be of the form  $\overline{\varphi\psi^k}$ , where  $\varphi$  is a finite-order character and  $k$  is an integer between 1 and  $p-1$ . Denote by  $L(\epsilon, s)$  the *primitive Hecke L-function* of  $\epsilon$ . Recall that the imprimitive (or partial)  $L$ -function differs from the primitive (or classical)  $L$ -function by a finite number of Euler factors. Let  $N_{K/\mathbb{Q}}$  denote the norm map. We can further define the *primitive algebraic Hecke L-value*,

$$L^{\text{alg}}(\overline{\varphi\psi^k}) = L^{\text{alg}}(\epsilon) := \frac{L(\epsilon, k)}{\Omega_\infty^k} = \frac{L(\overline{\varphi\psi^k} N_{K/\mathbb{Q}}^{-k}, 0)}{\Omega_\infty^k}.$$

Here,  $\Omega_\infty$  denotes a complex period for  $E/\mathbb{C}$ . Similarly, given an integral ideal  $\mathfrak{h}$  of  $K$ , we define the  $\mathfrak{h}$ -*imprimitive algebraic Hecke L-value*,

$$L_{\mathfrak{h}}^{\text{alg}}(\overline{\varphi\psi^k}) = L_{\mathfrak{h}}^{\text{alg}}(\epsilon) := \frac{L_{\mathfrak{h}}(\epsilon, k)}{\Omega_\infty^k} = \frac{L_{\mathfrak{h}}(\overline{\varphi\psi^k} N_{K/\mathbb{Q}}^{-k}, 0)}{\Omega_\infty^k}.$$

Note that  $L$  and  $L_{\mathfrak{h}}$  differ by the omission of the Euler factors at primes dividing  $\mathfrak{h}$ .

In what follows, we say that a Hecke character  $\epsilon$  of  $K$  is of *infinity type*  $(a, b)$  if its infinity component sends  $x$  to  $x^a \bar{x}^b$ . Under this convention,  $\psi$  has infinity type  $(-1, 0)$ , whereas the norm map  $N_{K/\mathbb{Q}}$  is of infinity type  $(-1, -1)$ . Thus, the Hecke character  $\overline{\psi^k} N_{K/\mathbb{Q}}^{-k}$  is of infinity type  $(k, 0)$ .

Henceforth, we fix a prime  $v \mid \mathfrak{p}$  of  $F$  and an embedding  $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}_p}$  so that  $v$  is sent into the maximal ideal  $\mathfrak{m}_{\overline{\mathbb{Q}_p}}$  of  $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ . This allows us to consider  $L_{\mathfrak{h}}^{\text{alg}}(\overline{\varphi\psi^k})$  as elements of  $\overline{\mathbb{Q}_p}$ . Throughout,  $\pi$  is a fixed uniformizer of  $F_v$  and we write  $\text{ord}_\pi$  for the valuation on  $\overline{\mathbb{Q}_p}$  normalized so that  $\text{ord}_\pi(\pi) = 1$ .

**Theorem 3.2** (Hida). *For all but finitely many characters  $\varphi$  that factor through  $\mathcal{R}(\mathfrak{g}q^\infty)^{\text{ac}}$ , we have*

$$\text{ord}_\pi \left( L_{(q)}^{\text{alg}}(\overline{\varphi\psi^k}) \right) = 0.$$

*Proof.* For each  $\varphi$ , we have  $\overline{\varphi} = \phi\eta$ , where  $\phi$  is a character of  $\Delta$  and  $\eta$  is a character of the Galois group  $\text{Gal}(\mathcal{R}(\mathfrak{g}q^\infty)^{\text{ac}}/F)$ . We may further decompose  $\phi$  into  $\phi'\nu^{-1}$ , where  $\nu$  is a character of  $\text{Gal}(F/\mathcal{R}(\mathfrak{g}))$  and  $\phi'$  is a character of  $\text{Gal}(\mathcal{R}(\mathfrak{g})/K)$ . We have the field diagram:

$$\begin{array}{c} \mathcal{R}(\mathfrak{g}q^\infty)^{\text{ac}} \\ \left. \begin{array}{c} \text{---} \eta \\ \text{---} \end{array} \right\} \mathbb{Z}_q \\ \mathcal{R}(\mathfrak{g}q) = F \\ \left. \begin{array}{c} \text{---} \nu^{-1} \\ \text{---} \phi' \end{array} \right\} \Delta \\ \mathcal{R}(\mathfrak{g}) \\ K \\ \downarrow \\ \mathbb{Q} \end{array}$$

We take the CM field  $M$  in [Hid07] to be the imaginary quadratic field  $K$ . We take the CM type  $\Sigma$  there to be the one that corresponds to the infinity type  $(1, 0)$  and  $\kappa = 0$ . Then the infinity type of the character  $\lambda$  in *op. cit.* becomes

$$k\Sigma + 0(1 - c) = k(1, 0) + (0, 0) - (0, 0) = (k, 0).$$

The condition (M1) in [Hid07, Theorem 4.3] does not hold since  $K/\mathbb{Q}$  is not unramified everywhere (it ramifies at the primes dividing the discriminant of  $K$ , which is nontrivial). Hence, we can apply the aforementioned theorem with  $\lambda$  and  $\chi^{-1}$  taken to be  $\overline{\psi^k} N^{-k} \phi'$  and  $\eta$ , respectively.  $\square$

*Remark 3.3* ([Lam14, proof of Theorem 3.1.9]). Let  $\mathfrak{g}$  be a fixed ideal as before. Fix an ideal  $\mathfrak{h}$  of  $\mathcal{O}_K$  which is coprime to  $\mathfrak{p}$  and divisible by  $\mathfrak{g}q$ . Recall that the  $\mathfrak{h}$ -imprimitive algebraic  $L$ -value of  $\overline{\varphi\psi^k}$  is given by

$$L_{\mathfrak{h}}^{\text{alg}}(\overline{\varphi\psi^k}) = \frac{L_{\mathfrak{h}}(\overline{\varphi\psi^k}, k)}{\Omega_{\infty}^k}.$$

Then, for almost all characters of  $\text{Gal}(\mathcal{R}(\mathfrak{g}q^{\infty})^{\text{ac}}/F) \cong \mathbb{Z}_q$ , we have that

$$\text{ord}_{\pi} \left( L_{(q)}^{\text{alg}}(\overline{\varphi\psi^k}) \right) = \text{ord}_{\pi} \left( L_{\mathfrak{h}}^{\text{alg}}(\overline{\varphi\psi^k}) \right).$$

This follows from the observation that for a given prime ideal  $\mathfrak{a}$  of  $K$  that is coprime to  $q$ , for almost all characters  $\eta$ ,

$$\text{ord}_{\pi} \left( 1 - \frac{\overline{\varphi\psi^k}(\mathfrak{a})}{(N\mathfrak{a})^k} \right) = 0$$

since  $\eta$  sends  $\mathfrak{a}$  to a  $q$ -power root of unity, and the images of  $q$ -power roots of unity under the reduction map on  $\mathcal{O}_{\mathbb{Q}_p}$  modulo  $\mathfrak{m}_{\mathbb{Q}_p}$  are distinct.

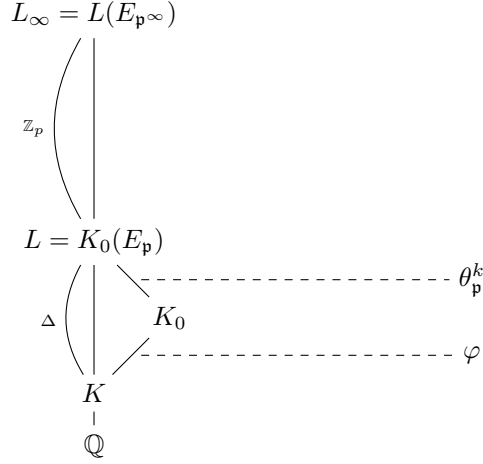
#### 4. CONSEQUENCES ON CLASS GROUPS

We now use Theorem 3.2 to study the growth of the  $p$ -part of the class group in an anticyclotomic  $\mathbb{Z}_q$ -extension. Let us introduce the necessary notation. Throughout,  $p \nmid 6q$  is a fixed prime that is split in  $K$  and  $E/K$  is a fixed CM elliptic curve as in the previous section (with Hecke character  $\psi$  whose conductor is  $\mathfrak{f}$ ). Let  $K_0$  be any finite abelian extension of  $K$  such that  $p$  is unramified in  $K_0$  and  $p \nmid [K_0 : K]$  (in the next subsection, we will let  $K_0$  vary inside the anticyclotomic tower  $\mathcal{R}(\mathfrak{g}q^{\infty})^{\text{ac}}$ ). Fix a prime  $\mathfrak{p}$  of  $K$  lying above  $p$ . Set  $L = K_0(E_{\mathfrak{p}})$  and  $L_{\infty} = L(E_{\mathfrak{p}^{\infty}})$ . Let  $\Delta = \text{Gal}(L/K)$  and  $\Gamma = \text{Gal}(L_{\infty}/L) \simeq \mathbb{Z}_p$ . Let  $\mathcal{G} = \text{Gal}(L_{\infty}/K) \cong \Delta \times \Gamma$  and  $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ .

Following [Rub91], we write  $\overline{C}(L_{\infty})$  (resp.  $U(L_{\infty})$ ) for the inverse limits over all finite sub-extensions inside  $L_{\infty}$  of the completion of the elliptic units (resp. local principal units) at  $\mathfrak{p}$ .

Fix an ideal  $\mathfrak{h}$  of  $\mathcal{O}_K$  which is coprime to  $\mathfrak{p}$ , is divisible by  $\mathfrak{f}$ , and is such that  $K_0 \subset K(E_{\mathfrak{h}}) = \mathcal{R}(\mathfrak{h})$ . Let  $\mu_K$  be the group of roots of unity of  $K$  and  $\lambda \in \mathcal{O}_K \setminus \mu_K$  such that  $\lambda \equiv 1 \pmod{\mathfrak{h}}$  with  $(\lambda, 6\mathfrak{h}\mathfrak{p}) = 1$ . We let  $\sigma_{(\lambda)} \in \text{Gal}(K_0/K)$  denote the Artin symbol associated to  $\lambda$ .

We further decompose  $\Delta$  as  $H \times I$ , where  $H = \text{Gal}(K_0/K)$  and  $I = \text{Gal}(K_0(E_{\mathfrak{p}})/K_0)$ . Here,  $I$  is the inertia subgroup at  $\mathfrak{p}$  inside  $\Delta$ . Let  $\theta_{\mathfrak{p}}$  denote the canonical character given by the Galois action on  $E_{\mathfrak{p}^{\infty}}$  restricted to  $I$ . Given a character  $\chi$  of  $\Delta$ , we write it as  $\varphi\theta_{\mathfrak{p}}^k$ , where  $\varphi$  is a character of  $H$  and  $1 \leq k \leq p - 1$ . We have the following diagram:



Before proceeding, we need to introduce the notion of an anomalous prime.

**Definition 4.1.** Fix a prime  $p$  and a number field  $\mathcal{K}$ . Let  $v$  be a prime above  $p$  in  $\mathcal{K}$  and write  $\kappa_v$  to denote the corresponding residue field. In the sense of Mazur (see [Maz72, p. 186]),  $E$  is *anomalous* at  $v$  if  $\tilde{E}(\kappa_v)[p] \neq 0$ .

Let  $w$  be a prime in  $L_\infty$  which lies above  $v$ . Denote by  $\mathcal{Z}$  the decomposition subgroup at  $\mathfrak{p}$  inside  $\mathcal{G}$ . Since  $\gcd(p, |\Delta|) = 1$ , the action of  $\Delta \cap \mathcal{Z}$  on  $\mu_{p^\infty}(L_{\infty, w}) = \mu_{p^\infty}$  gives a  $\mathbb{Z}_p$ -valued character which we denote by  $\chi_{\text{cyc}} : \Delta \cap \mathcal{Z} \rightarrow \mu_{p-1} \subseteq \mathbb{Z}_p^\times$ .

We now record a theorem of Lamplugh which will be important for our discussion.

**Theorem 4.2.** Let  $\chi = \varphi \theta_{\mathfrak{p}}^k$  be a character of  $\Delta$ . When  $E/K_0$  is anomalous at a prime above  $\mathfrak{p}$ , we assume that  $\chi|_{\Delta \cap \mathcal{Z}}$  is not the cyclotomic character. Let  $\mathfrak{h}$  and  $\lambda$  be as above. If

$$\text{ord}_\pi \left( \left( N(\lambda) - \lambda^k \varphi(\sigma(\lambda)) \right) \cdot L_{\mathfrak{h}}^{\text{alg}}(\overline{\varphi \psi^k}) \right) = 0,$$

then  $\overline{\mathcal{C}}(L_\infty)^\chi = U(L_\infty)^\chi$ . Here,  $M^\chi$  denotes the  $\chi$ -isotypic component of a  $\Lambda$ -module  $M$ .

*Proof.* See [Lam15, Theorem 7.7]. □

**4.1. Variations of class groups.** Let  $F = \mathcal{R}(\mathfrak{g}q)$  for some ideal  $\mathfrak{g}$  of  $\mathcal{O}_K$  such that  $\mathfrak{g}$  is divisible by  $\mathfrak{f}$ , is a product of primes that split in  $K$ ,  $p$  is unramified in  $F/K$ ,  $p \nmid [F : K]$ , and is coprime to  $\mathfrak{p}q$ . Furthermore, we assume that both  $\mathfrak{q}$  and  $\overline{\mathfrak{q}}$  are tamely ramified in  $F$ . Then  $\mathcal{R}(\mathfrak{g}q^\infty)^{\text{ac}}$  is a  $\mathbb{Z}_q$ -extension of  $F$ , and for integers  $n \geq 0$ , let  $F_n/F$  be the  $n$ -th layer of this  $\mathbb{Z}_q$ -extension. Note that only primes above  $q$  ramify in  $F_n/F$ ,  $p \nmid [F_n : K]$  (since  $q \neq p$ ), and  $F_n \subseteq \mathcal{R}(\mathfrak{g}q^{n+1})$ . Therefore, we may take  $K_0$  and  $\mathfrak{h}$  in the previous section to be  $F_n$  and  $\mathfrak{g}q^{n+1}$ , respectively.

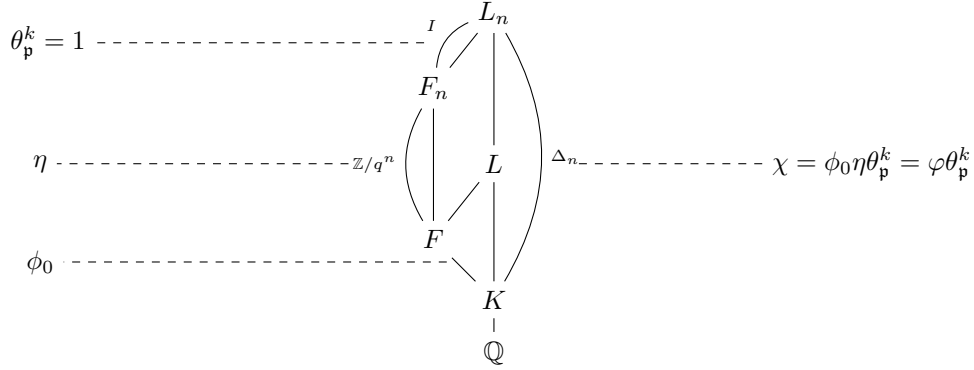
For  $n \geq 1$ , just as before we define  $L_n = F_n(E_{\mathfrak{p}})$ ,  $L_{n, \infty} = F_n(E_{\mathfrak{p}^\infty})$ ,  $\Delta_n = H_n \times I$ ,  $\mathcal{G}_n = \Delta_n \times \Gamma$ ,  $U_{n, \infty} = U(L_{n, \infty})$ , etc. As mentioned previously, we now let  $K_0$  vary inside the anticyclotomic tower  $\mathcal{R}(\mathfrak{g}q^\infty)^{\text{ac}}$ . Note that  $I = \text{Gal}(K_0(E_{\mathfrak{p}})/K_0) \cong \text{Gal}(L_n/F_n)$ . Define  $X_{n, \infty}$  to be the Galois group of the maximal abelian  $p$ -extension of  $L_{n, \infty}$  which is unramified outside  $\mathfrak{p}$ . By global class field theory we have the following four-term exact sequence

$$(1) \quad 0 \rightarrow \overline{\mathcal{E}}_{n, \infty} / \overline{\mathcal{C}}_{n, \infty} \rightarrow U_{n, \infty} / \overline{\mathcal{C}}_{n, \infty} \rightarrow X_{n, \infty} \rightarrow A_{n, \infty} \rightarrow 0.$$

Here,  $\overline{\mathcal{E}}_{n, \infty} = \overline{\mathcal{E}}(L_{n, \infty})$  is used to denote the global units of  $L_{n, \infty}$ . Finally,  $A_{n, \infty} = A(L_{n, \infty})$  is the inverse limit of the  $p$ -part of the class group for each finite extension of  $F_n$  contained inside  $L_{n, \infty}$ . In other words,  $A_{n, \infty}$  can be identified with the Galois group of the maximal abelian unramified  $p$ -extension of  $L_{n, \infty}$ . We now prove the key result which will be required in proving Theorem A.

**Theorem 4.3.** *There exists an integer  $N \geq 0$  such that  $X_{n,\infty}^I = X_{N,\infty}^I$  for all  $n \geq N$ , where  $M^I$  denotes the subgroup of  $M$  fixed by  $I$ .*

*Proof.* Let  $n \geq 0$  be an integer and consider a character  $\eta$  of  $\text{Gal}(F_n/F) \cong \mathbb{Z}/q^n$ . Let  $\phi_0$  be a character of  $\text{Gal}(F/K)$  and  $k$  an integer that is a multiple of  $p-1$  so that  $\theta_{\mathfrak{p}}^k$  is the trivial character. Set  $\varphi = \eta\phi_0$ . We draw the following field diagram for the convenience of the reader.



Let  $\mathcal{O}$  denote the ring of integers of the unique unramified  $\mathbb{Z}_q$ -extension of  $F_v$ . In other words,  $\mathcal{O} = \mathcal{O}_{F_v(\mu_{q^\infty})}$ . Let  $\lambda \in \mathcal{O}_K \setminus \mu_K$  such that  $\lambda \equiv 1 \pmod{\mathfrak{h}}$  and  $(\lambda, 6\mathfrak{h}\mathfrak{p}) = 1$  (where  $\mathfrak{h} = \mathfrak{g}q^{n+1}$ ). We have

$$\begin{aligned} (\lambda\bar{\lambda} - \lambda^k \varphi(\sigma(\lambda))) &\equiv 0 \pmod{\pi\mathcal{O}} \Leftrightarrow \varphi(\sigma(\lambda)) \equiv \bar{\lambda}\lambda^{1-k} \pmod{\pi\mathcal{O}} \\ &\Leftrightarrow \eta\phi_0(\sigma(\lambda)) \equiv \bar{\lambda}\lambda^{1-k} \pmod{\pi\mathcal{O}} \\ &\Leftrightarrow \eta(\sigma(\lambda)) \equiv \bar{\lambda}\lambda^{1-k}\phi_0^{-1}(\sigma(\lambda)) \pmod{\pi\mathcal{O}}. \end{aligned}$$

Note that  $\eta$  has exact order  $q^m$  for some  $m \geq 1$ . Therefore,  $\eta(\sigma(\lambda))$  is a primitive  $q^m$ -th root of unity. But in  $\mathcal{O}/\pi\mathcal{O}$ , the  $q$ -power roots of unity are distinct. Therefore, by the same argument outlined in Remark 3.3, there exists an integer  $N_1$  such that for all characters  $\eta$  of  $\text{Gal}(F_n/F)$  which do not factor through  $\Delta_{N_1}$  (with  $n \geq N_1$ ),

$$(2) \quad \text{ord}_\pi \left( N_{K/\mathbb{Q}}(\lambda) - \lambda^k \varphi(\sigma(\lambda)) \right) = 0.$$

By Theorem 3.2 and Remark 3.3, one can choose a sufficiently large  $N_2$  such that

$$(3) \quad \text{ord}_\pi \left( L_{\mathfrak{h}}^{\text{alg}}(\overline{\varphi\psi^k}) \right) = 0$$

for all characters  $\varphi$  of  $\Delta_n$  which do not factor through  $\Delta_{N_2}$  (with  $n \geq N_2$ ).

Set  $N = \max(N_1, N_2)$ . If  $\chi = \varphi$  is a character of  $\Delta_n$  which does not factor through  $\Delta_N$  (with  $n \geq N$ ), then (2) and (3) imply that

$$\text{ord}_\pi \left( \left( N_{K/\mathbb{Q}}(\lambda) - \lambda^k \varphi(\sigma(\lambda)) \right) \cdot L_{\mathfrak{h}}^{\text{alg}}(\overline{\varphi\psi^k}) \right) = 0.$$

Take  $K_0$  in Theorem 4.2 to be  $F_n$ . Since the restriction of the character  $\varphi$  to  $I$  is trivial, the hypothesis regarding  $E/K_0$  when a prime above  $\mathfrak{p}$  is an anomalous prime always holds. Therefore, we deduce that

$$U_{n,\infty}^\chi = \overline{\mathcal{C}}_{n,\infty}^\chi$$

for all characters  $\chi$  of  $\Delta_n$  that do not factor through  $\Delta_N$  with  $\chi|_I = 1$ . This implies

$$U_{n,\infty}^I / \overline{\mathcal{C}}_{n,\infty}^I = U_{N,\infty}^I / \overline{\mathcal{C}}_{N,\infty}^I.$$

Next, via the main conjecture of Iwasawa theory for imaginary quadratic fields (see [Rub91, Theorem 4.1(i)]) we can conclude that there exists an integer  $N \geq 0$  such that

$$\text{char}_\Lambda(X_{n,\infty}^I) = \text{char}_\Lambda(X_{N,\infty}^I)$$

for all  $n \geq N$ . Now, consider the restriction map

$$\pi_{n,N} : X_{n,\infty}^I \twoheadrightarrow X_{N,\infty}^I.$$

Since characteristic ideals are multiplicative in short exact sequences, the kernel of the above surjective map must be finite. However, a theorem of R. Greenberg (see [Gre78, Theorem §1]) ensures that there are no non-trivial finite submodules inside  $X_{n,\infty}^I$ . This forces the kernel to be trivial, i.e.,

$$X_{n,\infty}^I = X_{N,\infty}^I.$$

The proof of the theorem is now complete.  $\square$

We can now state and prove the auxiliary result that will allow us to conclude Theorem A. Our proof follows the proof of [Lam15, Theorem 7.10] very closely. We repeat the statement below for the convenience of the reader.

**Theorem 4.4.** *Let  $K$  be an imaginary quadratic field of class number 1. Let  $p$  and  $q$  be distinct primes ( $\geq 5$ ) which split in  $K$ . Let  $\mathfrak{g}$  be a fixed ideal of  $\mathcal{O}_K$  coprime to  $pq$  such that  $\mathfrak{g}$  is a product of split primes and is divisible by the conductor of an elliptic curve over  $K$  with CM by  $\mathcal{O}_K$ . Let  $F = \mathcal{R}(\tau\mathfrak{q})$ . We assume that  $p \nmid [F : K]$ . Let  $\mathcal{R}(\tau\mathfrak{q}^\infty)^{\text{ac}}/F$  denote the anticyclotomic  $\mathbb{Z}_q$ -extension and write  $F_n$  for the unique subextension of  $\mathcal{R}(\tau\mathfrak{q}^\infty)^{\text{ac}}/F$  whose degree is  $q^n$ . Then, there exists an integer  $N$  such that for all  $n \geq N$ ,*

$$\text{ord}_p(h(F_n)) = \text{ord}_p(h(F_N)).$$

*Proof.* Let the  $p$ -class group of  $F_n$  (resp.  $F_N$ ) be denoted by  $A(F_n)$  (resp.  $A(F_N)$ ). Since  $p$  does not divide  $[F_n : F_N]$ , we have an injection

$$(4) \quad A(F_N) \hookrightarrow A(F_n).$$

It follows from global class field theory that for all  $n \geq 0$ , we have the identification

$$A_{n,\infty} \simeq \text{Gal}(M_{n,\infty}/L_{n,\infty}),$$

where  $M_{n,\infty}$  is the maximal abelian unramified  $p$ -extension of  $L_{n,\infty}$ . Consider the following diagram

$$\begin{array}{ccc} A_{N,\infty}^I & \longrightarrow & A_{n,\infty}^I \\ \downarrow & & \downarrow \\ A(F_N) & \longrightarrow & A(F_n) \end{array}$$

where the vertical maps are given by restriction and are surjective because the extension  $L_{n,\infty}/F_n$  and  $L_{N,\infty}/F_N$  are totally ramified at primes above  $\mathfrak{p}$ . Furthermore, the top horizontal map is surjective by Theorem 4.3 and the exact sequence (1). Therefore, the bottom row is a surjective map as well. When combined with (4), we see that the bottom row is in fact an isomorphism. This completes the proof of the theorem.  $\square$

The following lemma allows us to complete the proof of Theorem A via Theorem 4.4.

**Lemma 4.5.** *Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be ideals of  $\mathcal{O}_K$ . If  $p \nmid [\mathcal{R}(\mathfrak{a}) : K] \cdot [\mathcal{R}(\mathfrak{b}) : K]$ , then  $p \nmid [\mathcal{R}(\text{lcm}(\mathfrak{a}, \mathfrak{b})) : K]$ .*

*Proof.* Let us write  $\mathfrak{a} = \prod \mathfrak{p}_i^{m_i}$ ,  $\mathfrak{b} = \prod \mathfrak{p}_i^{n_i}$ , where  $\mathfrak{p}_i$  are distinct prime ideals of  $\mathcal{O}_K$ . Recall that  $K$  is of class number 1. By the theory of complex multiplication, if  $I$  is an ideal of  $\mathcal{O}_K$ , we have

$$\text{Gal}(\mathcal{R}(I)/K) = \text{Gal}(K(E[I])/K) \cong (\mathcal{O}_K/I)^\times.$$

Thus, by the Chinese remainder theorem,

$$p \nmid |(\mathcal{O}_K/\mathfrak{p}_i^{m_i})^\times|, \quad p \nmid |(\mathcal{O}_K/\mathfrak{p}_i^{n_i})^\times|$$



for all  $i$ . As  $\text{lcm}(\mathfrak{a}, \mathfrak{b}) = \prod \mathfrak{p}_i^{\max(m_i, n_i)}$ , we deduce that

$$p \nmid |(\mathcal{O}_K / \text{lcm}(\mathfrak{a}, \mathfrak{b}))^\times| = [\mathcal{R}(\text{lcm}(\mathfrak{a}, \mathfrak{b})) : K]. \quad \square$$

We can now prove Theorem A from the introduction.

**Theorem.** *Let  $K$  be an imaginary quadratic field of class number 1. Let  $p$  and  $q$  be distinct primes ( $\geq 5$ ) which split in  $K$ . Let  $\mathfrak{r}$  be a fixed ideal of  $\mathcal{O}_K$  coprime to  $pq$  such that  $\mathfrak{r}$  is a product of split primes. Let  $\mathcal{F} = \mathcal{R}(\mathfrak{r}q)$  and write  $\mathcal{R}(\mathfrak{r}q^\infty)^{\text{ac}}/\mathcal{F}$  for the anticyclotomic  $\mathbb{Z}_q$ -extension. Assume that  $p \nmid [\mathcal{F} : K]$ . Then, there exists an integer  $N$  such that for all  $n \geq N$ ,*

$$\text{ord}_p(h(\mathcal{F}_n)) = \text{ord}_p(h(\mathcal{F}_N)).$$

*Proof.* Let  $E/K$  be a CM elliptic curve of conductor  $\mathfrak{f}$  such that all the prime divisors of  $\mathfrak{f}$  are split in  $K$  but the prime divisors (which are  $\geq 5$ ) of  $[\mathcal{R}(\mathfrak{f}) : K]$  are not split in  $K$ . Such elliptic curves exist as we have seen in §2.

Let  $\mathfrak{r}$  be any ideal of  $\mathcal{O}_K$  and  $p, q$  be two distinct primes satisfying the hypotheses in the statement of the theorem. Set  $\mathfrak{g} = \text{lcm}(\mathfrak{f}, \mathfrak{r})$  and define  $F = \mathcal{R}(\mathfrak{g}q)$ . By assumption,  $p \nmid [\mathcal{R}(\mathfrak{r}q) : K]$  and we have chosen our auxiliary CM elliptic curve so that  $p \nmid [\mathcal{R}(\mathfrak{f}) : K]$ . Thus, it follows from Lemma 4.5 that  $p \nmid [\mathcal{R}(\mathfrak{g}q) : K]$ . Furthermore, both  $\mathfrak{f}$  and  $\mathfrak{r}$  are only divisible by split primes. Therefore, Theorem 4.4 holds for the ideal  $\mathfrak{g}$ .

Since  $p \nmid [F_n : \mathcal{F}_n]$  and  $p \nmid [\mathcal{F}_{n+1} : \mathcal{F}_n]$  for all  $n \geq 0$ , we have

$$A(\mathcal{F}_n) \hookrightarrow A(F_n), \quad A(\mathcal{F}_n) \hookrightarrow A(\mathcal{F}_{n+1}).$$

Theorem 4.4 asserts that  $\text{ord}_p(h(F_n))$  stabilizes as  $n \rightarrow \infty$ . Hence, the same is true for  $\text{ord}_p(h(\mathcal{F}_n))$ .  $\square$

## 5. ASYMPTOTIC GROWTH OF FINE SELMER GROUPS OF ABELIAN VARIETIES

**5.1. Definition of fine Selmer groups.** Suppose  $F$  is a number field. Throughout,  $A/F$  is a fixed abelian variety. We fix a finite set  $S$  of primes of  $F$  containing  $p$ , the primes dividing the conductor of  $A$ , as well as the Archimedean primes. We write  $S_f$  to denote the set of finite primes. Denote by  $F_S$ , the maximal algebraic extension of  $F$  unramified outside  $S$ . For every (possibly infinite) extension  $L$  of  $F$  contained in  $F_S$ , write  $G_S(L) = \text{Gal}(F_S/L)$ . Write  $S(L)$  for the set of primes of  $L$  above  $S$ . If  $L$  is a finite extension of  $F$  and  $w$  is a place of  $L$ , we write  $L_w$  for its completion at  $w$ ; when  $L/F$  is infinite, it is the union of completions of all finite sub-extensions of  $L$ .

**Definition 5.1.** Let  $L/F$  be an algebraic extension. The  $p$ -primary fine Selmer group of  $A$  over  $L$  is defined as

$$\text{Sel}_0(A/L) = \ker \left( H^1(G_S(L), A[p^\infty]) \rightarrow \bigoplus_{v \in S} H^1(L_v, A[p^\infty]) \right).$$

Similarly, the  $p$ -fine Selmer group of  $A$  over  $L$  is defined as

$$\text{Sel}_0(A[p]/L) = \ker \left( H^1(G_S(L), A[p]) \rightarrow \bigoplus_{v \in S} H^1(L_v, A[p]) \right).$$

Note that  $\text{Sel}_0(A/L)$  is independent of the choice of  $S$ , whereas the definition of  $\text{Sel}_0(A[p]/L)$  depends on  $S$ ; see for example [LM16, Lemma 4.1 and p. 86]. Since our main result concerns  $\text{Sel}_0(A/L)$ , we suppress  $S$  from the notation of  $\text{Sel}_0(A[p]/L)$  for simplicity.

It is easy to observe that if  $F_\infty/F$  is an infinite extension,

$$\text{Sel}_0(A/F_\infty) = \varinjlim_L \text{Sel}_0(A/L), \quad \text{Sel}_0(A[p]/F_\infty) = \varinjlim_L \text{Sel}_0(A[p]/L),$$

where the inductive limits are taken with respect to the restriction maps and  $L$  runs over all finite extensions of  $F$  contained in  $F_\infty$ . Next, we define the notion of  $p$ -rank of an abelian group  $G$ .

**Definition 5.2.** Let  $G$  be an abelian group. Define the  $p$ -rank of  $G$  as

$$r_p(G) = r_p(G[p]) := \dim_{\mathbb{F}_p}(G[p]).$$

**5.2. Growth of fine Selmer groups in  $\mathbb{Z}_q$ -extensions.** In this section, we prove the following theorem which essentially says that the  $p$ -part of the class group and the  $p$ -primary fine Selmer group have similar growth behaviour in  $\mathbb{Z}_q$ -extensions. Our result is motivated by [LM16, Section 5].

**Theorem 5.3.** *Let  $A$  be a  $d$ -dimensional abelian variety defined over a number field  $F$ . Let  $S(F)$  be a finite set of primes in  $F$  consisting precisely of the primes above  $q$ , the primes of bad reduction of  $A$ , and the Archimedean primes. Let  $F_\infty/F$  be a fixed  $\mathbb{Z}_q$  extension such that primes in  $S_f(F)$  are finitely decomposed in  $F_\infty/F$  and suppose  $[F_n : F] = q^n$ . Further suppose that  $A[p] \subseteq A(F)$ . Then as  $n \rightarrow \infty$ ,*

$$\left| r_p(\text{Sel}_0(A/F_n)) - 2dr_p(\text{Cl}(F_n)) \right| = O(1).$$

If  $A[p] \subseteq A(F)$ , then the action of  $G_F$  on  $A[p]$  is trivial. Let  $A^\vee$  be the dual abelian variety. The action on the dual representation,  $A^\vee[p]$  is also trivial. This tells us that  $A^\vee[p] \subseteq A^\vee(F)$ . Therefore, Theorem 5.3 allows us to deduce the following result.

**Corollary 5.4.** *With the same hypothesis as in Theorem 5.3*

$$\left| r_p(\text{Sel}_0(A/F_n)) - r_p(\text{Sel}_0(A^\vee/F_n)) \right| = O(1).$$

To prove Theorem 5.3, we need a few lemmas.

**Lemma 5.5.** *Consider the following short exact sequence of co-finitely generated abelian groups*

$$P \rightarrow Q \rightarrow R \rightarrow S.$$

Then,

$$\left| r_p(Q) - r_p(R) \right| \leq 2r_p(P) + r_p(S).$$

*Proof.* See [LM16, Lemma 3.2]. □

**Lemma 5.6.** *Let  $F_\infty$  be any  $\mathbb{Z}_q$ -extension of  $F$  such that all the primes in  $S_f(F)$  are finitely decomposed. Let  $F_n$  be the subfield of  $F_\infty$  such that  $[F_n : F] = q^n$ . Then*

$$\left| r_p(\text{Cl}(F_n)) - r_p(\text{Cl}_S(F_n)) \right| = O(1).$$

*Proof.* For each  $F_n$ , we write  $S_f(F_n)$  for the set of finite primes of  $F_n$  above  $S_f$ . For each  $n$ , we have the following exact sequence

$$\mathbb{Z}^{|S_f(F_n)|} \longrightarrow \text{Cl}(F_n) \xrightarrow{\alpha_n} \text{Cl}_S(F_n) \longrightarrow 0$$

(see [NSW08, Lemma 10.3.12]). Since the class group is always finite, it follows that  $\ker(\alpha_n)$  is finite. Also,  $r_p(\ker(\alpha_n)) \leq |S_f(F_n)|$  and  $r_p(\ker(\alpha_n)/p) \leq |S_f(F_n)|$ . By Lemma 5.5,

$$\left| r_p(\text{Cl}(F_n)) - r_p(\text{Cl}_S(F_n)) \right| \leq 2|S_f(F_n)| = O(1). \quad \square$$

**Lemma 5.7.** *Let  $F_\infty/F$  be a  $\mathbb{Z}_q$ -extension and let  $F_n$  be the subfield of  $F_\infty$  such that  $[F_n : F] = q^n$ . Let  $A$  be an abelian variety defined over  $F$ . Suppose that all primes of  $S_f(F)$  are finitely decomposed in  $F_\infty/F$ . Then*

$$\left| r_p(\text{Sel}_0(A[p]/F_n)) - r_p(\text{Sel}_0(A/F_n)) \right| = O(1).$$

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Sel}_0(A[p]/F_n) & \rightarrow & H^1(G_S(F_n), A[p]) & \rightarrow & \bigoplus_{v \in S(F_n)} H^1(F_{n,v_n}, A[p]) \\ & & \downarrow s_n & & \downarrow f_n & & \downarrow \gamma_n \\ 0 & \rightarrow & \text{Sel}_0(A/F_n)[p] & \rightarrow & H^1(G_S(F_n), A[p^\infty])[p] & \rightarrow & \bigoplus_{v \in S(F_n)} H^1(F_{n,v_n}, A[p^\infty])[p]. \end{array}$$

Both  $f_n$  and  $\gamma_n$  are surjective. Since  $A[p] \subset A(F_n)$ , the kernel of these maps are given by

$$\begin{aligned} \ker(f_n) &= A(F_n)[p^\infty]/p \simeq (\mathbb{Z}/p\mathbb{Z})^{2d}, \\ \ker(\gamma_n) &= \bigoplus_{v_n \in S(F_n)} A(F_{n,v_n})[p^\infty]/p \simeq \bigoplus_{v_n \in S_f(F_n)} (\mathbb{Z}/p\mathbb{Z})^{2d}, \end{aligned}$$

where the last isomorphism follows from our assumption that  $p$  is odd.

Observe that  $r_p(\ker(s_n)) \leq r_p(\ker(f_n)) = 2d$  and that  $r_p(\ker(\gamma_n)) = 2d|S_f(F_n)|$ . By hypothesis,  $S_f(F_n)$  is bounded as  $n$  varies. It follows from the snake lemma that both  $r_p(\ker(s_n))$  and  $r_p(\text{coker}(s_n))$  are finite and bounded. Applying Lemma 5.5 to the following exact sequence

$$0 \rightarrow \ker(s_n) \rightarrow \text{Sel}_0(A[p]/F_n) \rightarrow \text{Sel}_0(A/F_n)[p] \rightarrow \text{coker}(s_n) \rightarrow 0$$

completes the proof.  $\square$

*Proof of Theorem 5.3.* By hypothesis,  $A[p] \subseteq A(F)$ . Therefore,  $A[p] \simeq (\mathbb{Z}/p)^{2d}$ . We have

$$H^1(G_S(F_n), A[p]) = \text{Hom}(G_S(F_n), A[p]).$$

There are similar identifications for the local cohomology groups. Thus,

$$\text{Sel}_0(A[p]/F_n) \simeq \text{Hom}(\text{Cl}_S(F_n), A[p]) \simeq \text{Cl}_S(F_n)[p]^{2d}$$

as abelian groups. Therefore,

$$r_p(\text{Sel}_0(A[p]/F_n)) = 2dr_p(\text{Cl}_S(F_n)).$$

The theorem now follows from Lemmas 5.6 and 5.7.  $\square$

Let  $p^{e_n}$  be the largest power of  $p$  that divides the class number of  $F_n$ . If  $e_n$  is bounded then it follows (trivially) that the  $p$ -rank is bounded. Thus, the following corollary is immediate.

**Corollary 5.8.** *Let  $p \neq q$ . Let  $F/\mathbb{Q}$  be any finite extension of  $\mathbb{Q}$  and  $F_\infty/F$  be any  $\mathbb{Z}_q$ -extension of  $F$ . Let  $p^{e_n}$  be the exact power of  $p$  dividing the class number of the  $n$ -th intermediate field  $F_n$ . Let  $A/F$  be an abelian variety such that  $A[p] \subseteq A(F)$ . If  $e_n$  is bounded as  $n \rightarrow \infty$ , then  $r_p(\text{Sel}_0(A/F_n))$  is bounded independently of  $n$ .*

In addition to Theorem A, there are some other results in the literature where it is known that the  $p$ -part of the class group stabilizes in a  $\mathbb{Z}_q$ -extension (when  $p, q$  are distinct primes). These were discussed briefly in the introduction and are recorded here more precisely.

- (1) ([Was78, Theorem]) Let  $F/\mathbb{Q}$  be an abelian extension of  $\mathbb{Q}$  and  $F_\infty/F$  be the cyclotomic  $\mathbb{Z}_q$ -extension of  $F$ . If  $p^{e_n}$  be the exact power of  $p$  dividing the class number of the  $n$ -th intermediate field  $F_n$ , then  $e_n$  is bounded as  $n \rightarrow \infty$ .
- (2) ([Lam15, Theorem 7.10]) Let  $p, q$  be fixed odd distinct primes both  $\geq 5$ ,  $K$  be an imaginary quadratic field of class number 1 where  $p$  and  $q$  split, and  $E/K$  be an elliptic curves with CM by  $\mathcal{O}_K$  and good reduction at  $p, q$ . Let  $K_\infty$  be the  $\mathbb{Z}_q$  extensions of  $K$  which is unramified outside  $\mathfrak{q}$  (resp.  $\bar{\mathfrak{q}}$ ). Let  $\mathfrak{g}$  be a fixed ideal of  $\mathcal{O}_K$  such that it is coprime to  $pq$  and  $F = \mathcal{R}(\mathfrak{g}\mathfrak{q})$  is of degree prime-to- $p$  over  $F$ . Then, the  $p$ -part of the class number stabilizes in  $FK_\infty = \mathcal{R}(\mathfrak{g}\mathfrak{q}^\infty)$ . However, since  $p$  is assumed to be unramified in  $F$  in *loc. cit.*, the hypothesis  $A[p] \subseteq A(F)$  in Theorem 5.3 is unlikely to hold. The same can be said regarding the setting studied in Theorem A.

**Theorem 5.9.** *With notation as above, suppose that the  $p$ -rank of the fine Selmer group, denoted by  $r_p(\text{Sel}_0(A/F_n))$  stabilizes in a  $\mathbb{Z}_q$ -extension of  $F$ . Then there exists  $n \geq 0$ , such that for all  $m \geq n$ , the restriction map induces an isomorphism*

$$\text{Sel}_0(A/F_n) \simeq \text{Sel}_0(A/F_m).$$

*Proof.* The following argument is similar to the one presented in [Lam14, p. 15], where instead of classical Selmer groups, we consider fine Selmer groups. Consider the extension  $F_m/F_n$ . Then  $[F_m : F_n] = q^{m-n} = t$  (say). The restriction map

$$\text{Gal}(\overline{F}/F_n) \longrightarrow \text{Gal}(\overline{F}/F_m)$$

induces the restriction homomorphism

$$\text{res} : \text{Sel}_0(A/F_n) \longrightarrow \text{Sel}_0(A/F_m).$$

Since  $\gcd(q, p) = 1$ , this map is an injection. Moreover, we have

$$\text{Sel}_0(A/F_n) \xrightarrow{\text{res}} \text{Sel}_0(A/F_m) \xrightarrow{\text{cores}} \text{Sel}_0(A/F_n) \xrightarrow{t^{-1}} \text{Sel}_0(A/F_n)$$

where  $\text{cores} \circ \text{res} = t$ . The composition  $\text{res} \circ \text{cores} \circ t^{-1}$  is the identity map; thus, the exact sequence

$$0 \longrightarrow \text{Sel}_0(A/F_n) \longrightarrow \text{Sel}_0(A/F_m) \longrightarrow \text{Sel}_0(A/F_m)/\text{Sel}_0(A/F_n) \longrightarrow 0$$

is split exact.

Let us write  $\text{Sel}_0(A/F_n) = (\mathbb{Q}_p/\mathbb{Z}_p)^{s_n} \oplus T_n$ , where  $s_n \geq 0$  and  $T_n$  is a finite  $p$ -group. Then,

$$r_p(\text{Sel}_0(A/F_n)) = s_n + r_p(T_n).$$

The injection  $\text{Sel}_0(A/F_n) \hookrightarrow \text{Sel}_0(A/F_m)$  tells us that  $s_m \geq s_n$ . If the  $p$ -rank  $r_p(\text{Sel}_0(A/F_n))$  eventually stabilizes it follows that  $s_n$  also stabilizes. Denote the cokernel of the injection by  $C_{m,n}$ . By duality, we have the short exact sequence

$$0 \rightarrow C_{m,n}^\vee \rightarrow \mathbb{Z}_p^{s_m} \oplus T_m^\vee \rightarrow \mathbb{Z}_p^{s_n} \oplus T_n^\vee \rightarrow 0.$$

When  $s_m = s_n$ ,  $C_{m,n}^\vee$  must be finite. Consequently, the image of  $C_{m,n}^\vee$  in  $\text{Sel}_0(A/F_n)^\vee$  is contained inside  $T_n^\vee$ . Furthermore, since the short exact sequence splits, we deduce the isomorphism

$$T_m = T_n \oplus C_{m,n}.$$

As  $s_n$  stabilizes,  $r_p(T_n)$  also stabilizes. Therefore,  $C_{m,n}$  has to be 0 eventually.  $\square$

Theorem B is now an immediate corollary of Theorems 5.3 and 5.9.

**Corollary 5.10.** *Let  $p, q$  be distinct odd primes. Let  $F$  be any number field and  $A/F$  be an abelian variety such that  $A[p] \subseteq A(F)$ . Let  $F_\infty/F$  be a  $\mathbb{Z}_q$ -extension where the primes above  $q$  and the primes of bad reduction of  $A$  are finitely decomposed. If the  $p$ -part of the class group stabilizes, i.e., there exists  $N \geq 0$  such that for all  $n \geq N$ ,*

$$\text{ord}_p(h(F_n)) = \text{ord}_p(h(F_N)),$$

*then the growth of the  $p$ -primary fine Selmer group stabilizes in the  $\mathbb{Z}_q$ -extension as well, i.e., there exists an integer  $N' \geq N$  such that for all  $n \geq N'$ , the restriction map induces an isomorphism*

$$\text{Sel}_0(A/F_n) \simeq \text{Sel}_0(A/F_{N'}).$$

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