Katz definition of a modular form f of wt k over R.

is a function $f : \{(E, \iota_N, \omega)\} \to R$

E/R = elliptic curve

$\iota_N = \Gamma_1(N)$ structure $(N \in R^\times)$

$\iota_N : \mathbb{Z}/N \to E[N]$

$\omega \neq 0$ invariant differential on E

st $f(E, \iota_N, \lambda\omega) = \lambda^{-k} f(E, \iota_N, \omega) \quad \lambda \in R^\times$

f is invariant for isomorphism

f is stable by base change $\forall h : R \to S$

$f(E, \iota_N, \omega) = h(f(E, \iota_N, \omega))$

Let $z \in \mathfrak{h}$, $\mathbb{Z} + z\mathbb{Z} \subseteq \mathbb{C}$ be a lattice

$E_z = \mathbb{C}/_{\mathbb{Z}+z\mathbb{Z}}$ = elliptic curve $/ \mathbb{C}$

$1/N \in \mathbb{C}/_{\mathbb{Z}+z\mathbb{Z}}[N]$

$u \in \mathbb{C}$, du be an inv. diff'l on E_z

given a Katz modular form of wt $k / R \hookrightarrow \mathbb{C}$

$f^{an}(z) = f(E_z, 1/N, du)$

Let $\gamma \in \Gamma_1(N)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$E_{\gamma z} = \mathbb{C}/_{\mathbb{Z} + \frac{az+b}{cz+d}\mathbb{Z}} \underset{\text{homothetic}}{\sim} \mathbb{C}/_{(cz+d)\mathbb{Z} + (az+b)\mathbb{Z}}$

$= \mathbb{C}/_{\mathbb{Z}+z\mathbb{Z}}$

$du_{\gamma z} = (cz+d)\, du_z$

$f(\gamma z) = f(E_{\gamma z}, 1/N, du_{\gamma z}) = f(E_z, 1/N, (cz+d)\, du_z)$

$= f(z)(cz+d)^k$

$\Gamma_1(N)\backslash\mathfrak{h} = Y \subseteq \bar{X}$ compactification

$$\begin{tikzcd} \mathbb{C}\times\mathfrak{h} \arrow[d] & & \mathcal{E} \arrow[d] & \mathcal{T} = \mathcal{I}som(\mathcal{O}_Y, \Omega^1_{\mathcal{E}/X}) \arrow[l] \\ \mathfrak{h} \arrow[rr, squiggly] & & Y & \end{tikzcd}$$

$\mathbb{C}\times\mathfrak{h} \circlearrowleft SL_2(\mathbb{Z})$

$\mathcal{T}$: $\mathbb{G}_m$ - torsor; $\Omega^1_{\mathcal{E}/X}$: free of rk 1

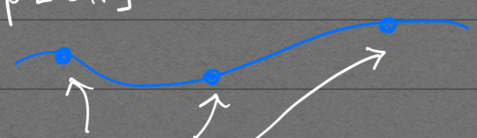
$\omega^k = \mathcal{O}_{\mathcal{T}}[-k] \to$ of wt k for the action of $\mathbb{G}_m$

$M_k = H^0(X, \omega^k)$

$\{\text{holo at } \infty\} \leftrightarrow f(T = \text{Tate curve}, \iota, du) \in R[[q]]$

fix pt N, $p \geq 5$, $\mathcal{O} = \mathbb{Z}_p[\xi_N]$

$X/\mathcal{O} \xrightarrow[\text{redn mod } p]{} X/\mathbb{F}_q$



supersingular pts

$E/\mathbb{F}_q$ is ordinary $\#(E[p](\bar{\mathbb{F}}_p)) = p$

supersingular $\#(E[p](\bar{\mathbb{F}}_p)) = 1$

fact: (for ordinary)

$$1 \to \mu_{p^\infty} \to E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0 \quad (\text{over } \bar{\mathbb{F}}_p)$$

Define X_1^{ord} = ord locus of $X/\mathbb{F}_q$

X_n^{ord} = counter image of X_1^{ord} in $X_n = X/\mathcal{O}_{p^n}$.

$$\begin{tikzcd} & X_n^{ord} \\ (\mathbb{Z}/p^n)^\times \arrow[r, phantom] & T_{m,n} = \mathrm{Isom}((E/X_n^{ord}[p^m])^0, \mu_{p^m}) \arrow[u] \arrow[u, bend left] \end{tikzcd}$$

$T_{\infty,n} = \varprojlim_m T_{m,n}$; $T = \varprojlim_n T_{\infty,n}$ (Igusa tower)

$\mathbb{Z}_p^\times$ torsor $\quad T \downarrow X^{ord}$ = formal scheme $\mathrm{Spf}(\mathcal{O}) = \varinjlim_n X_n^{ord}$

Defn: p-adic modular forms (functions on Igusa tower)

$V = H^0(X^{ord}, \mathcal{O}_T)$

fact: $M_k = H^0(X, \omega^k) \hookrightarrow V$

$\bigoplus_k M_k \hookrightarrow V$ and are p-adically dense } Imp

$\mathbb{G}_m$-torsor $\rightsquigarrow \mathbb{Z}_p^\times = \mathbb{G}_m(\mathbb{Z}_p)$-torsor

$\mathcal{O}_{\mathcal{T}} \hookrightarrow \mathcal{O}_T$ alg funcns on $\mathbb{G}_m$ define analytic func on $\mathbb{Z}_p^\times$

This implies that $V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{M}(\mathbb{Q}_p)$ ← from today morning

f on X^{ord}, $\|f\|_{X^{ord}} = |f(q)|_p$

Let $f \in M_k \longrightarrow f \in \mathcal{O}_T$

$f(E, \iota_N, \gamma : E[p^\infty]^\circ \xrightarrow{\sim} \mu_{p^\infty}) = ?\ f(E, \iota_N, \gamma^* \frac{du}{u})$

$\mu_{p^\infty} \hookrightarrow \mathbb{G}_m$

$du/u \longleftarrow du/u$ canon inv diff'l

by restriction

$M_k^{p\text{-adic}} = V[-k]$ for the action of $\mathbb{Z}_p^\times$

$\uparrow$ I can take any $k : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$

M_k

We can define $e := \lim_n U_p^{n!} \curvearrowright V$

Ordinary family

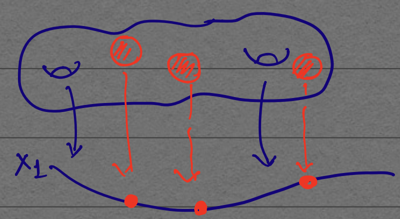
$eV =$ fin rank module $\mathcal{O}[[\mathbb{Z}_p^\times]]$

↑ Hida

An ordinary form is a p-adic eigenform of slope 0 for $U = U_p$

The problem is that U_p is not compact [f = any eigenform $\lambda \in p\mathbb{Z}_p \rightsquigarrow f_\lambda$ st $U_p f_\lambda = \lambda f_\lambda$]

X^{rig} over $Sp(\mathrm{Frac}(\mathcal{O}))$



X_1^{ord} = non-vanishing locus of A = Hasse-invariant

$$\begin{tikzcd} E/\bar{\mathbb{F}}_p \arrow[rr, "E[p]"] \arrow[dr, bend right, "\mathrm{Frob}"'] & & E \\ & E^{(p)} \arrow[ur, bend right, "\text{(Verschebung)}"'] & \end{tikzcd}$$

defines a modular form of wt $p-1$.

Ha = a lift of H_n Hasse-inv mod p.

$$v_p(Ha) : X^{rig} \longrightarrow [0,1]$$
$$x \longmapsto \mathrm{Truncate}\left(v_p(Ha(x))\right)$$

f = a p-adic modular form, generally don't extend to X^{rig} because they have poles in the supersingular disc

Katz: $v \in [0,1)$

$$X(v) = \left\{ x \in X^{rig} \mid 0 \leq v_p(Ha) \leq v \right\}$$

$X(0) = (X^{ord})^{rig}$

Define $M_{k,v} \subseteq M_k^{p\text{-adic}} = \bigcup M_{k+(p-1)\ell} / E_{p-1}^{\ell}$ Hasse inv.

$$M_{k,v} = \{ p\text{-adic modular form of wt } k, \text{ st } f \text{ extends to } X(v) \}$$

functorial defn: f overconvergent form of wt k is

$$f : \{ (E, \iota_N, \omega, Y) \} \to R$$

y = wt $1-p$ modular form st $y\, Ha = p^v$

Classically: U_p is defined via a correspondence

$$C_p = \{(E, H, C \subseteq E[p] \text{ st } H \cap C = 0)\}$$

$$\begin{tikzcd} & C_p \arrow[dl] \arrow[dr] & \\ X_0(p) & & X_0(p) \end{tikzcd}$$

$X_0(p) = \{(E, H \subseteq E[p])\}$

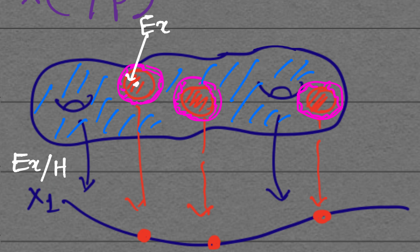
$X_0(p) = \{(E/C, H/C)\}$

$$U_p(f)(E,H) = \sum_{H \cap C = 0} f(E/C, H/C)$$

Thm (Lubin-Tate) Let $v < \frac{p}{p+1}$, $E_x \in X(v)$. Then

(1) $\exists!\ H \subseteq E_x[p]$ st H lifts the ker(Frob) on E_x/p

(2) $E_x/H \in X(v/p)$



Prop: U_p defines a map, if $v \leq 1/p+1$

$$U_p : M_{k,v} \longrightarrow M_{k,vp}$$

Pf: Let $E_x \in X(vp)$, we want to evaluate U_p & $E_x = \sum_{H \cap C = 0} f(E_x/C)$

If $v_p\, Ha(x) < vp < p/p+1$

$v_p\, Ha(y) < v$

y be a pt corr to E_x/H $\to$ & choose $f(E_x/C)$

Thm: $U_p : M_{k,v} \to M_{k,vp} \xrightarrow{Res} M_{k,v}$ is compact if $v < \frac{1}{p+1}$

Pf: Restriction of functions is compact!

Proof of Lubin / Katz / Coleman: $[p]$ on the formal gp of an elliptic curve

$$[p] = pX + a_p X^p + \sum_{m \geq 2} c_m X^{m(p-1)+1}$$

$\equiv$ Hasse inv mod p

Calculate the zeroes of the series

$$H = \mathrm{Spec}\left(\frac{A[X]}{X^p + p/H_a X}\right)$$

$\parallel$