GROWTH OF FINE SELMER GROUPS IN INFINITE TOWERS

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Abstract. In this paper, we study the growth of fine Selmer groups in two cases. First, we study the growth of fine Selmer ranks in multiple $\mathbb{Z}_p$-extensions. We show that the growth of the fine Selmer group is unbounded in such towers. We recover a sufficient condition to prove the $\mu = 0$ Conjecture for cyclotomic $\mathbb{Z}_p$-extensions. We show that in certain non-cyclotomic $\mathbb{Z}_p$-towers, the μ-invariant of the fine Selmer group can be arbitrarily large. Second, we show that in an unramified p-class field tower, the growth of the fine Selmer group is unbounded. This tower is non-Abelian and non p-adic analytic.

1. Introduction

Let E be an elliptic curve defined over a number field F. The Mordell-Weil theorem says that the group of rational points of E, denoted $E(F)$, is finitely generated. Selmer groups play a crucial role in studying the rational points on elliptic curves. In [21], Mazur introduced the Iwasawa theory of Selmer groups. He described the growth of the p-primary part of the Selmer group in $\mathbb{Z}_p$-towers and showed that in the *ordinary* case the growth is controlled.

In [6], Coates and Sujatha introduced the study of a certain subgroup of the Selmer group, called the fine Selmer group. They showed that these subgroups have stronger finiteness properties than the classical Selmer group.

In the 1980's in a series of papers, Cuoco and Monsky studied the growth of class groups in $\mathbb{Z}_p^d$-towers when $d > 1$. In Section 3, we study the growth of the p-rank of fine Selmer group in multiple $\mathbb{Z}_p$-extensions. We prove this growth is unbounded in general. When $d = 1$, we recover a sufficient condition to prove the Classical $\mu = 0$ Conjecture. In a specific $\mathbb{Z}_p^2$-tower, it is possible to provide an asymptotic formula. In Section 4 we study the growth of fine Selmer groups in non-cyclotomic $\mathbb{Z}_p$-extensions. We prove an analogue of Iwasawa's theorem [12, Theorem 1] for fine Selmer groups and show that there exist non-cyclotomic $\mathbb{Z}_p$-extensions where the fine Selmer group can have arbitrarily large μ-invariant.

Greenberg had developed a general theory to study the growth of the p-primary component of the Selmer group in *any* p-adic analytic Galois extension. In [24], Murty and Ouyang studied the growth of $\mathfrak{p}$-Selmer ranks for Abelian varieties with CM, in p-class field tower of F. By the work of Boston [3] and Hajir [9], we know that this tower defines an infinite Galois extension of F whose Galois group is not p-adic analytic. Hence, growth in these towers is not covered by the general theory developed by Greenberg. Murty-Ouyang showed that the Selmer rank in such a

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tower is unbounded [24]. We prove that in fact the fine Selmer rank in such a tower is unbounded.

2. Preliminaries

Let $M \xrightarrow{\phi} N$ be a homomorphism of Abelian groups (modules etc.), we denote by $M[\phi]$ the kernel of ϕ.

Definition 2.1. *For an Abelian group G, its p-rank $r_p(G) := \dim_{\mathbb{Z}/p\mathbb{Z}} G[p]$.*

2.1. **Review of Fine Selmer Group.** Throughout this article p will denote an odd prime. Let F be a number field and A/F be an Abelian variety. Let $S = S(F)$ be a finite set of primes in F, which contains the Archimedean primes (denoted S_∞), the primes of bad reduction of E (denoted S_{bad}) and the primes above p (denoted S_p). Denote by F_S the maximal algebraic extension of F unramified outside S and set $G_S(F) = \mathrm{Gal}(F_S/F)$. Define the p-fine Selmer group $R_p^S(A/F)$ and p^∞-fine Selmer group $R_{p^\infty}(A/F)$ as in [31].

Let K be an imaginary quadratic field and p be a rational prime that splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in K. Let F/K be any finite Galois extension and E be an elliptic curve defined over it.

Define the $\mathfrak{p}$-fine Selmer group as

$$R_{\mathfrak{p}}^S(E/F) := \ker\left(H^1\left(G_S(F),\ E[\mathfrak{p}]\right) \to \bigoplus_{v\in S} H^1\left(F_v,\ E[\mathfrak{p}]\right)\right).$$

Analogously, define $R_{\mathfrak{p}^n}^S(E/F)$ for any $n \geq 1$. Taking direct limits, define the $\mathfrak{p}^\infty$-fine Selmer group (also called the $\mathfrak{p}$-primary fine Selmer group),

$$R_{\mathfrak{p}^\infty}(E/F) := \varinjlim_m R_{\mathfrak{p}^m}^S(E/F)$$

where the limit is with respect to the maps induced by $E[\mathfrak{p}^m] \hookrightarrow E[\mathfrak{p}^{m+1}]$. An equivalent definition is the following,

$$R_{\mathfrak{p}^\infty}(E/F) = \ker\left(H^1(G_S(F),\ E[\mathfrak{p}^\infty]) \to \bigoplus_{v\in S} H^1(F_v,\ E[\mathfrak{p}^\infty])\right).$$

Though it appears a priori that the definition depends on S, one can check [17, Lemma 4.1] that the $\mathfrak{p}^\infty$-fine Selmer group is independent of this choice. The same is true for the p^∞-fine Selmer group. There are analogous definitions for every finite extension, F'/F. Suppose F_∞/F is contained in F_S with $\mathrm{Gal}(F_\infty/F)$ being a p-adic Lie group. Set

$$R_{\mathfrak{p}^\infty}(E/F_\infty) := \varinjlim R_{\mathfrak{p}^\infty}(A/F').$$

where the inductive limit is over finite extensions of F contained in F_∞.

3. Growth of p-Fine Selmer Ranks in $\mathbb{Z}_p^d$-Extensions

In this section, we study the growth of fine Selmer groups in $\mathbb{Z}_p^d$-extensions of a number field. These are Abelian pro-p, p-adic analytic extensions. In Theorem 3.3, we obtain an estimate for the rank growth of fine Selmer groups in a specific $\mathbb{Z}_p^2$ extension. In the general setting, we can only prove that the p-rank growth is unbounded under certain assumptions (see Theorem 3.8). Our result is a refinement of the result of Lim-Murty [16, Proposition 5.1].

3.1. **Notation:** Consider a $\mathbb{Z}_p^d$-extension F_∞/F. Set $\Sigma = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^d$. Further, write $\Sigma_n = \Sigma^{p^n}$ and $F_n = F_\infty^{\Sigma_n}$. Here, F_n is a $(\mathbb{Z}/p^n\mathbb{Z})^d$-extension of F inside F_∞. The Iwasawa algebra Λ_d, is the completed group ring of Σ over $\mathbb{Z}_p$; let $\overline{\Lambda}_d$ be the completion (with respect to the powers of the augmentation ideal) of the $\mathbb{Z}/p\mathbb{Z}$ group ring of Σ. We have,

$$\Lambda_d \simeq \mathbb{Z}_p[[T_1, T_2, \ldots, T_d]]; \quad \overline{\Lambda}_d \simeq \mathbb{Z}/p\mathbb{Z}[[T_1, T_2, \ldots, T_d]].$$

Let $M(F_\infty)$ (resp. $M_S(F_\infty)$) be the maximal Abelian unramified pro-p extension of F_∞ (resp. the maximal Abelian unramified pro-p extension of F_∞ such that primes in S are totally split). The Iwasawa-Greenberg module (resp. S-Iwasawa-Greenberg module) is denoted by $X := \mathrm{Gal}(M(F_\infty)/F_\infty)$ (resp. $X_S := \mathrm{Gal}(M_S(F_\infty)/F_\infty)$). Let $\overline{X}$ (resp. $\overline{X_S}$) be the reduction mod p of the Iwasawa-Greenberg module (resp. S-Iwasawa-Greenberg module). Define a (resp. a') to be the height of the local ring $\overline{\Lambda}_d/\operatorname{Ann}\overline{X}$ (resp. $\overline{\Lambda}_d/\operatorname{Ann}\overline{X_S}$); so $1 \leq a, a' \leq d$.

3.2. **Trivialising Extension of an Elliptic Curves with CM.** Let K be an imaginary quadratic field. Suppose further $p \neq 2, 3$ splits as $\mathfrak{p}\bar{\mathfrak{p}}$ in K. Let E/K be an elliptic curve with good reduction at p and CM by $\mathcal{O}_K$. Set

$$F = K(E[p]),\ F_\infty = K(E[p^\infty]),\ G = \mathrm{Gal}(F_\infty/F),\ \mathcal{G}_\infty = \mathrm{Gal}(F_\infty/K).$$

Remark 3.1. The hypotheses, $\mathrm{End}_F(E)$ is the maximal order of K, involves no real loss of generality since every E/F with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_F(E)$ isomorphic to K is isogenous over F to one with this property.

We refer to F_∞ as the *trivializing extension of* F. By the Weil pairing, F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension of F, which in turn is denoted by F_{cyc}.

The group G is pro-p and is isomorphic to $\mathbb{Z}_p^2$. By the theory of complex multiplication, the Galois group $\mathrm{Gal}(F/K)$ is Abelian. Since p splits completely in $K/\mathbb{Q}$, the order of $\mathrm{Gal}(F/K)$ is coprime to p. It then follows that $\mathcal{G}_\infty \simeq G \times \Delta$ where $\Delta \simeq \mathrm{Gal}(F/K)$.

Proposition 3.2. *Let E be an elliptic curve over the imaginary quadratic field K with CM by $\mathcal{O}_K$. Set $F = K(E[p])$ and let F_∞ be the trivialising extension. Then*

$$\left| r_p\left(R_{p^\infty}(E/F_n)\right) - 2r_p\left(\mathrm{Cl}(F_n)\right)\right| = O(1)$$

where $F_n = K(E[p^n])$ is a $(\mathbb{Z}/p^n\mathbb{Z})^2$-extension of F contained in F_∞ and $\mathrm{Cl}(F_n)$ is the class group of F_n.

Proof. The proof is identical to that of [17, Theorem 5.1] provided the finite primes $S_f(F)$ are finitely decomposed in F_∞/F. Recall that primes above p ramify, and all ramified primes in this extension are finitely decomposed.

By hypothesis, E attains good reduction at all the primes of F [28, Lemma 2] or [30]. The definition of $R_{p^\infty}(E/F_n)$ is independent of S, so choose S to be $S_p \cup S_\infty$. The condition of finite decomposition is met [7, Page 45], and the proposition follows. $\square$

Using the main result in [23], there is the following interesting corollary.

Theorem 3.3. *Let E be an elliptic curve over an imaginary quadratic field K, with CM by $\mathcal{O}_K$. Set $F = K(E[p])$ and F_∞ be the trivialising extension. Then for*

F_n and a as defined before,

$$r_p\left(R_{p^\infty}\left(E/F_n\right)\right) = 2cp^{an} + O(p^{(a-1)n}).$$

When $a=1$, c is an integer and when $a=2$, $c = \operatorname{rank}_{\overline{\Lambda}_2}(\overline{X})$.

Proof. Consider [23, Theorem 1.9] when $d=2$. Monsky proves that

$$r_p\left(\operatorname{Cl}(F_n)\right) = cp^{an} + O\left(p^{(a-1)n}\right).$$

When $a=1$, $c\in\mathbb{Z}$; when $a=2$, $c=\operatorname{rank}_{\overline{\Lambda}_2}(\overline{X})$. The theorem follows. □

3.3. **General $\mathbb{Z}_p^d$-Extension.** Using a variant of Monsky's result in [23], we show that the fine Selmer rank growth is unbounded in a general $\mathbb{Z}_p^d$-extension.

Lemma 3.4. *[17, Lemma 3.2] Let G be any pro-p group and M be a discrete G-module that is cofinitely generated over $\mathbb{Z}_p$. Define $h_1(G) := r_p\left(\left(H^1(G,\mathbb{Z}/p\mathbb{Z})\right)\right)$. If $h_1(G)$ is finite then $r_p\left((H^1(G,M))\right)$ is finite. Furthermore*

$$\begin{aligned} h_1(G)r_p(M^G) - r_p\left((M/M^G)^G\right) &\leq r_p\left(H^1(G,M)\right) \\ &\leq h_1(G)\left(\operatorname{corank}_{\mathbb{Z}_p}(M) + \log_p\left(\left|M/M_{\mathrm{div}}\right|\right)\right). \end{aligned}$$

Lemma 3.5. *Let A be a d-dimensional Abelian variety defined over a number field F. Consider a finite set of primes S containing $S_p\cup S_{bad}\cup S_\infty$. Suppose $A(F)[p]\neq 0$. Then*

$$r_p\left(R_{p^\infty}\left(A/F\right)\right) \geq r_p\left(\operatorname{Cl}_S\left(F\right)\right) r_p\left(A\left(F\right)[p]\right) - 2d \tag{1}$$

$$r_p\left(R_p^S\left(A/F\right)\right) \geq r_p\left(\operatorname{Cl}_S\left(F\right)\right) r_p\left(A\left(F\right)[p]\right) - 2d. \tag{2}$$

Proof. Inequality 1 is proven in [17, Lemma 4.3]. We therefore provide a proof of Inequality 2. The proof is similar but we include it for the sake of completeness.

Let $H_S(F)=H$ be the p-Hilbert S-class field of F. Denote the Galois group, $\operatorname{Gal}(H_S(F)/F) = \operatorname{Cl}_S(F)$ by $\mathcal{G}$. We call $\operatorname{Cl}_S(F)$ the S-class group of F. We have

$$\begin{aligned} r_p\left(\operatorname{Cl}_S(F)\right) &= r_p\left(\operatorname{Cl}_S(F)/p\operatorname{Cl}_S(F)\right) \\ &= \dim_{\mathbb{Z}/p\mathbb{Z}}\left(H^1(\mathcal{G},\ \mathbb{Z}/p\mathbb{Z})\right) \\ &= h_1(\mathcal{G}). \end{aligned}$$

The first equality follows from finiteness of the S-class group. The second equality is the definition of the S-class group. By Lemma 3.4, to prove Inequality 2, it suffices to prove

$$r_p\left(R_p^S\left(A/F\right)\right) \geq r_p\left(H^1\left(\mathcal{G},\ A(H)[p]\right)\right). \tag{3}$$

To prove this inequality, consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_p^S(A/F) & \longrightarrow & H^1(G_S(F),\ A[p]) & \longrightarrow & \bigoplus_{v\in S} H^1(F_v,\ A[p]) \\ & & \big\downarrow \alpha & & \big\downarrow \beta & & \big\downarrow \gamma \\ 0 & \longrightarrow & R_p^S(A/H) & \longrightarrow & H^1(G_S(H),\ A[p]) & \longrightarrow & \bigoplus_{v\in S}\bigoplus_{w|v} H^1(H_w,\ A[p]) \end{array}$$

The vertical maps are given by restriction maps. Write $\gamma = \oplus_v \gamma_v$ where

$$\gamma_v : H^1\left(F_v,\ A[p]\right) \to \bigoplus_{w|v} H^1\left(H_w,\ A[p]\right).$$

By the inflation-restriction sequence, $\ker \gamma_v = H^1(\mathcal{G}_v,\ A[p])$ where $\mathcal{G}_v$ is the decomposition group of $\mathcal{G}$ at v. By the definition of the p-Hilbert S-class field, all the primes in $S(F)$ split completely in H_S; hence $\mathcal{G}_v = 1$. Thus, $\ker \gamma$ is trivial.

By inflation-restriction, $\ker \beta = H^1\left(\mathcal{G},\ A(H)[p]\right)$. By a diagram chasing argument,

$$H^1\left(\mathcal{G},\ A\left(H\right)[p]\right) \hookrightarrow R_p^S(A/F).$$

This implies

$$r_p\left(R_p^S(A/F)\right) \geq r_p\left(H^1\left(\mathcal{G},\ A\left(H\right)[p]\right)\right).$$

The result follows. □

Remark 3.6. Lemma 3.5 holds at every layer F_n/F in the $\mathbb{Z}_p^d$-tower.

Repeating the argument as in the original paper of Monsky, it is possible to prove the following variant of [23, Theorem 1.9].

Theorem 3.7. *With notation introduced at the start of the section, there is a positive real constant c, such that*

$$r_p\left(\mathrm{Cl}_S(F_n)\right) = cp^{a'n} + O\left(p^{(a'-1)n}\right),$$

where $\mathrm{Cl}_S(F_n)$ is the S-class group of F_n.

The fact that $c > 0$, follows from [22, Corollary to Theorem 1.8].

Theorem 3.8. *Let A be an Abelian variety defined over F, and let p be an odd prime such that $A(F)[p] \neq 0$. Let F_∞/F be a $\mathbb{Z}_p^d$-extension such that $\overline{X_S}$ is infinite. Let F_n be a $(\mathbb{Z}/p^n\mathbb{Z})^d$-extension of F inside F_∞. Then as $n \to \infty$, $r_p\left(R_{p^\infty}(A/F_n)\right) \to \infty$.*

Proof. By Theorem 3.7, if $\overline{X_S}$ is infinite then $r_p\left(\mathrm{Cl}_S(F_n)\right)$ approaches infinity as n approaches infinity. The conclusion follows from Lemma 3.5. □

Remark 3.9. Theorem 3.8 is a fine Selmer variant of [16, Proposition 5.1]. Unlike the result of Lim-Murty, we do not need to impose any restrictions on the reduction type at p. This comes at a cost that we need a smaller group $\overline{X_S}$ to be infinite.

3.4. **Growth in $\mathbb{Z}_p$-Extensions.** We focus on the case $d = 1$. Let F be a number field and F_∞/F be *any* $\mathbb{Z}_p$-extension with $\Gamma = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$. There is an isomorphism $\Lambda(\Gamma) \simeq \mathbb{Z}_p[[T]]$ given by $\gamma - 1 \mapsto T$, where γ is a topological generator of Γ. By the structure theorem for a finitely generated $\Lambda(\Gamma)$-module M, there is a pseudo-isomorphism

$$M \to \Lambda(\Gamma)^r \oplus \bigoplus_{i=1}^{s} \Lambda(\Gamma)\Big/\left(p^{m_i}\right) \oplus \bigoplus_{j=1}^{t} \Lambda(\Gamma)\Big/\left(f_j^{l_j}\right)$$

where $s,\ t$ are finite, $m_i,\ l_j > 0$, and each f_j is a distinguished polynomial. If $r = 0$, define $\mu(M) := \sum_{i=1}^{s} m_i$.

Conjecture (Classical $\mu = 0$ Conjecture [11]). *Let F_{cyc}/F be the cyclotomic $\mathbb{Z}_p$-extension and $X(F_{\mathrm{cyc}})$ be the associated Iwasawa module. Then $\mu(X(F_{\mathrm{cyc}})) = 0$.*

Remark 3.10. It is known that $\mu\left(X\left(F_{\mathrm{cyc}}\right)\right) = \mu\left(X_S\left(F_{\mathrm{cyc}}\right)\right)$ [25, Corollary 11.3.6].

Lemma 3.11. *[16, Lemma 5.3] Let M be a finitely generated $\Lambda(\Gamma)$-module and $w_n = (1+T)^{p^n} - 1$. For $n \gg 0$,*

$$r_p\left(M \big/ (p, w_n)\, M\right) = \big(r(M) + s(M)\big)\, p^n + O(1).$$

The Pontryagin dual of the p^∞-fine Selmer group is denoted $Y(A/F_\infty)$. Also,

$$Y(A/F_\infty) \big/ pY(A/F_\infty) \simeq \big(R_{p^\infty}(A/F_\infty)[p]\big)^\vee .$$

We prove a simple variant of Inequality 1 , which follows from the next lemma.

Lemma 3.12. *[17, Lemma 5.2] Let F_∞ be any $\mathbb{Z}_p$ extension of F such that all the primes above p are finitely decomposed. Let F_n be the subfield of F_∞ such that $[F_n : F] = p^n$. Let $S = S_p \cup S_\infty$. Then*

$$\left| r_p\left(\mathrm{Cl}(F_n)\right) - r_p\left(\mathrm{Cl}_S(F_n)\right) \right| = O(1).$$

Proof. The only non-Archimedean primes in S are the primes above p, they are finitely decomposed in any F_∞/F. □

Proposition 3.13. *Let A/F be an Abelian variety and p be an odd prime. Suppose A has good reduction everywhere over F and $A(F)[p] \neq 0$. Let F_∞ be a $\mathbb{Z}_p$-extension of F and F_n be the subfield of F_∞ such that $[F_n : F] = p^n$. Then*

$$r_p\left(R_{p^\infty}(A/F_n)\right) \geq r_p\left(\mathrm{Cl}(F_n)\right) r_p\left(A(F_n)[p]\right) + O(1) \tag{4}$$

Proof. The definition of $R_{p^\infty}(A/F_n)$ is independent of S, choose $S = S(F_n)$ to be precisely the set of Archimedean primes and the primes above p. The proposition follows from Lemma 3.5 upon noting that Lemma 3.12 applies in this case. □

Using Proposition 3.13 we give the fine Selmer variant of a theorem of Lim-Murty [16, Theorem 5.6] and Česnavičius [4, Proposition 7.1].

Theorem 3.14. *Let $p \neq 2$. Let A be an Abelian variety defined over F with good reduction everywhere over F and $A(F)[p] \neq 0$. Let F_∞ be any $\mathbb{Z}_p$-extension of F. Then*

$$r\left(Y(A/F_\infty)\right) + s\left(Y(A/F_\infty)\right) \geq s\left(X(F_\infty)\right) r_p\left(A\left(F_\infty\right)[p]\right)$$

where $X(F_\infty)$ is the Iwasawa module over the $\mathbb{Z}_p$-extension F_∞/F.

Proof. Set $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$. Consider the following commutative diagram with the vertical maps given by restriction

$$\begin{array}{ccccccc}
0 & \to & R(A/F_n) & \to & H^1(G_S(F_n),\ A[p^\infty]) & \to & \bigoplus_{v_n} H^1(F_{n,v_n},\ A[p^\infty]) \\
 & & \big\downarrow r_n & & \big\downarrow f_n & & \big\downarrow \gamma_n \\
0 & \to & R(A/F_\infty)^{\Gamma_n} & \to & H^1(G_S(F_\infty),\ A[p^\infty])^{\Gamma_n} & \to & \left(\varinjlim_n \bigoplus_{v_n} H^1\left(F_{n,v_n},\ A[p^\infty]\right)\right)^{\Gamma_n}
\end{array}$$

Note that $r_p\left(\ker(f_n)\right) \leq 2d$. Thus $\ker(r_n)$ has bounded p-rank. The Control Theorem for fine Selmer groups guarantee that $\mathrm{coker}(r_n)$ is also finite and bounded independent of n (see [29, chapter VII] and [31]). Further,

$$\begin{aligned}\Big(r\left(Y(A/F_\infty)\right)+s\left(Y(A/F_\infty)\right)\Big)p^n &= r_p\Big(R_{p^\infty}\left(A/F_n\right)\Big)+O(1)\\ &\geq r_p\left(\mathrm{Cl}(F_n)\right)r_p\left(A(F_n)[p]\right)+O(1).\end{aligned}$$

The first equality follows from Lemma 3.11 upon observing that the Pontryagin dual of $R_{p^\infty}(A/F_\infty)^{\Gamma_n}[p]$ is equal to $Y/(p,w_n)Y$. The second inequality follows from Proposition 3.13. There exists n_0, such that for $n\geq n_0$

$$\mathrm{Cl}(F)\Big/p\,\mathrm{Cl}(F)\to X(F_\infty)\Big/\left(p,\tfrac{w_n}{w_{n_0}}\right)\to 0.$$

The kernel of this map is bounded independent of n [25, Lemma 11.1.5]. Since $X(F_\infty)$ is *always* a finitely generated *torsion* $\Lambda(\Gamma)$-module (see [25, Proposition 11.3.1]), by Lemma 3.11

$$r_p\left(\mathrm{Cl}(F_n)\right)=s(X(F_\infty))p^n+O(1).$$

The result follows since $r_p(A(F_\infty)[p])=r_p(A(F_n)[p])$ for sufficiently large n. $\square$

Recall that in the cyclotomic $\mathbb{Z}_p$-extension, all the primes are finitely decomposed. Hence, Lemma 3.12 holds for all finite sets $S\supseteq S_p\cup S_{bad}\cup S_\infty$. In this case, we may drop the hypothesis of *good reduction everywhere* from Theorem 3.14. to obtain

$$r\left(Y(A/F_\infty)\right)+s\left(Y(A/F_\infty)\right)\geq s\left(X_S(F_\infty)\right)r_p\left(A(F_\infty)[p]\right).$$

An immediate corollary is the following.

Corollary 3.15. *Let $p\neq 2$. If there exists one elliptic curve E/F with $E(F)[p]\neq 0$ such that $r\Big(Y\left(E/F_{\mathrm{cyc}}\right)\Big)=s\Big(Y\left(E/F_{\mathrm{cyc}}\right)\Big)=0$, then the Classical Iwasawa $\mu=0$ Conjecture holds for F_{cyc}/F.*

We remind the readers that $r\Big(Y\left(E/F_{\mathrm{cyc}}\right)\Big)=s\Big(Y\left(E/F_{\mathrm{cyc}}\right)\Big)=0$ is precisely Coates-Sujatha Conjecture A [6]. The above corollary is true if we consider Abelian varieties in general, but we formulate it in this way since the conjecture of Coates-Sujatha is for elliptic curves.

Remark 3.16. Corollary 3.15 is an effective variant of known results in the literature (see [6, Theorem 3.4], [17, Theorem 5.4]). In [1], Aribam had used [6, Theorem 3.4] to give new evidence of the Iwasawa $\mu=0$ Conjecture.

4. Arbitrarily Large μ-Invariants of Fine Selmer Groups in Non-Cyclotomic $\mathbb{Z}_p$-Extensions

We begin by recalling a theorem proven by Iwasawa [12, Theorem 1].

Theorem 4.1 (Iwasawa's Theorem for Non-Cyclotomic Extensions)**.** *Let F be the cyclotomic field of p-th or 4-th roots of unity according as $p>2$ or $p=2$. For any given integer $N\geq 1$, there exists a cyclic extension L/F of degree p and a $\mathbb{Z}_p$-extension L_∞/L such that*

$$\mu\left(X\left(L_\infty\right)\right)\geq N.$$

The close relationship between fine Selmer groups and class groups in $\mathbb{Z}_p$-towers raises the following natural question: can the μ-invariant associated with a fine Selmer group be arbitrarily large in a non-cyclotomic $\mathbb{Z}_p$-extension. We answer this question in the affirmative under the following hypothesis which we will call the (universal) analogue of the Weak Leopoldt Conjecture.

Hypothesis: Let K be any number field and A be an Abelian variety defined over K. Let p be an odd prime and K_∞/K be any $\mathbb{Z}_p$-extension of K. Then

$$H^2\left(G_S\left(K_\infty\right),\ A[p^\infty]\right) = 0.$$

Theorem 4.2. *Let F be the cyclotomic field of p-th roots of unity for $p > 2$. Let A/F be an Abelian variety of dimension d such that $A(F)[p] \neq 0$. Suppose the (universal) analogue of the Weak Leopoldt Conjecture holds. Given an integer $N \geq 1$, there exists a cyclic Galois extension L/F of degree p and a $\mathbb{Z}_p$-extension L_∞/L such that*

$$\mu\Big(Y\left(A/L_\infty\right)\Big) \geq N.$$

Remark 4.3.

1. It is believed that for all number fields and all $\mathbb{Z}_p$-extensions F_∞/F, the cohomology group $H^2\left(G_S(F_\infty),\ A[p^\infty]\right) = 0$. This is often called the Abelian variety analogue of the Weak Leopoldt Conjecture over F. This implies $Y(A/F_\infty)$ is $\Lambda(\Gamma)$-torsion where $\Gamma = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$ (see [20, Theorem 2.2] or [6, Lemma 3.1] for the case when primes in S are finitely decomposed). Thus, conjecturally $r\left(Y(A/F_\infty)\right)$ should be 0. When $F_\infty = F_{\mathrm{cyc}}$, this is known in a large number of cases by a deep result of Kato [13] For the $\mathbb{Z}_p$-extension unramified outside $\mathfrak{p}$, it was proved in [27] (see also [15]). For non-cyclotomic extensions, there is little unconditional evidence towards this conjecture (see [2]).
2. In view of recent results of Hajir-Maire [10], we believe it might be possible to extend the above theorem to more general situations. It would be interesting to consider analogues of results of Ozaki [26, Theorem 2] for p^∞-fine Selmer groups.

4.1. **Review of Iwasawa's Result.** In proving Theorem 4.1, Iwasawa crucially used a result of Chevalley [5] on *ambiguous class number formula.*

Definition 4.4. *Let F be a number field and L/F be a cyclic $\mathbb{Z}/p\mathbb{Z}$-extension. Let $G = \mathrm{Gal}(L/F) = \langle\sigma\rangle$. An ideal class $[\mathfrak{a}] \in \mathrm{Cl}(L)$ is called*

- ***ambiguous*** *if $[\mathfrak{a}]^\sigma = [\mathfrak{a}]$, i.e. there exists an element $\alpha \in L^\times$ such that $\mathfrak{a}^{\sigma-1} = (\alpha)$.*
- ***strongly ambiguous*** *if $\mathfrak{a}^{\sigma-1} = (1)$.*

The subgroup of the class group $\mathrm{Cl}(L)$ consisting of ambiguous ideal classes (resp. strongly ambiguous ideal classes) is denoted by $\mathrm{Am}(L/F)$ (resp. $\mathrm{Am}_{\mathrm{st}}(L/F)$).

Theorem (Ambiguous Class Number Formula)**.** *The number of ambiguous ideal classes is given by*

$$\#\,\mathrm{Am}(L/F) = h(F) \times \frac{p^{T-1}}{[E_F : E_F \cap NL^\times]} \tag{5}$$

$$\#\,\mathrm{Am}_{\mathrm{st}}(L/F) = h(F) \times \frac{p^{T-1}}{[E_F : NE_L]} \tag{6}$$

where $h(F)$ is the class number of the base field F, T is the number of ramified primes, E_F is the unit group of F, and $E_F \cap NL^\times$ is the subgroup of units that are norms of elements of L. Moreover, the above two formulas are equivalent.

Proof. See [5] or [14, Theorem 1]. □

The following variant of the ambiguous class number formula can be proved very similar to [14, Theorem 1].

Proposition 4.5. *Let F be a number field and L/F be a cyclic $\mathbb{Z}/p\mathbb{Z}$-extension with σ a generator of the Galois group $G = \mathrm{Gal}(L/F)$. Let D be the degree of the extension $L/\mathbb{Q}$ and T be the number of primes of F that ramify in L. Then,*

$$r_p\left(\mathrm{Am}_{\mathrm{st}}\left(L/F\right)\right) \geq T - D.$$

4.2. **Proof of Theorem 4.2.** The proof can be divided into four main steps. The first three are the same as that of Theorem 4.1.

Step 1: Let F_+ be the maximal totally real subfield of F. Let F_{nc}/F be a non-cyclotomic $\mathbb{Z}_p$-extension which satisfies a special property: if F_n^{nc} is the n-th layer of the $\mathbb{Z}_p$-tower, then F_n^{nc} is Galois over F_+. Furthermore, $G_n = \mathrm{Gal}(F_n^{\mathrm{nc}}/F_+)$ is the dihedral group of order $2p^n$. Such a non-cyclotomic $\mathbb{Z}_p$-extension exists by Iwasawa's construction [12, Section 2].

Step 2: Let $\mathfrak{l}_+ \nmid p$ be a prime ideal of F_+ which is inert in F, and be the unique prime ideal of F above $\mathfrak{l}_+$. Note $\mathfrak{l}_+$ is unramified in F_n^{nc}. Using group theoretic properties of the dihedral group and class field theory, it can be shown that $\mathfrak{l}$ is *totally split* in F_n^{nc} [12]. This holds for every n, therefore $\mathfrak{l}$ is totally split in F_{nc}/F. By Chebotarev density theorem, there are infinitely many prime ideals $\mathfrak{l}_+$ in F_+ which are inert in F. Thus, there are infinitely many prime ideals in F which split completely in F_{nc}/F.

Step 3: Choose prime ideals $\mathfrak{l}_1, \dots, \mathfrak{l}_t$, $t \geq 1$, in F which are prime to p and are totally split in F_{nc}/F. We know from Step 2 that there are infinitely many such primes. Let η be a non-zero element of F which is divisible exactly by the first power of $\mathfrak{l}_i$ for $1 \leq i \leq t$. Set

$$L = F\left(\sqrt[p]{\eta}\right); \qquad L_\infty = LF_{\mathrm{nc}}.$$

Note $F_{\mathrm{nc}} \cap L = F$ and L_∞/L is a $\mathbb{Z}_p$-extension. Let L_n be the n-th layer of the $\mathbb{Z}_p$ tower L_∞/L, then L_n/F_n is a cyclic extension of degree p. More precisely,

$$L_n = F_n^{\mathrm{nc}}\left(\sqrt[p]{\eta}\right), \qquad n \geq 0. \tag{7}$$

We need to prove a technical lemma. Recall that the definition of the p-primary fine Selmer group is independent of the choice of S. Denote the subset of finite primes of S be S_f. Further, set $\left|S_f\right| = s_0$.

Lemma 4.6. *Let F be the cyclotomic field of p-th roots of unity for $p > 2$. Let A/F be an Abelian variety of dimension d such that $A(F)[p] \neq 0$. Suppose the analogue of the weak Leopoldt conjecture holds. Let L/F be a $\mathbb{Z}/p\mathbb{Z}$-extension as constructed in Theorem 4.1. Then,*

$$r_p\left(R_{p^\infty}(A/L_n)\right) \geq r_p\left(A\left(L_n\right)[p]\right)\left(s\left(X(L_\infty)\right)p^n\right) - 4ds_0p^n + c$$

where L_n is as defined in (7) and c is a constant.

Proof. The proof follows the same steps as Proposition 3.13. But primes in the $\mathbb{Z}_p$-extension L_∞/L need not be finitely decomposed, so the analysis is more intricate.

Step a: Consider the following short exact sequence for all n [25, Lemma 10.3.12],

$$\mathbb{Z}^{|S_f(L_n)|} \to \mathrm{Cl}(L_n) \xrightarrow{\alpha_n} \mathrm{Cl}_S(L_n) \to 0.$$

Comparing p-ranks in this short exact sequence [17, Lemma 3.2], we obtain

$$\Big| r_p\left(\mathrm{Cl}\left(L_n\right)\right) - r_p\left(\mathrm{Cl}_S\left(L_n\right)\right) \Big| \leq 2\big|S_f(L_n)\big| \leq 2s_0p^n.$$

Step b: Lemma 3.5 applied to the number field L_n yields

$$r_p\left(R_{p^\infty}\left(A/L_n\right)\right) \geq r_p\left(\mathrm{Cl}_S\left(L_n\right)\right) r_p\left(A\left(L_n\right)[p]\right) - 2d.$$

Therefore, we obtain

$$\begin{aligned} r_p\left(R_{p^\infty}\left(A/L_n\right)\right) &\geq r_p\left(\mathrm{Cl}\left(L_n\right) - 2s_0p^n\right) r_p\left(A\left(L_n\right)[p]\right) - 2d \\ &\geq r_p\left(\mathrm{Cl}\left(L_n\right)\right) r_p\left(A\left(L_n\right)[p]\right) - 4ds_0p^n - 2d. \end{aligned}$$

By the Structure Theorem (in particular Lemma 3.11),

$$r_p\left(\mathrm{Cl}\left(L_n\right)\right) = s\left(X(L_\infty)\right)p^n + O(1).$$

Plugging this back into the above inequality, proves the lemma. □

Step 4: We are assuming that the analogue of the weak Leopoldt Conjecture holds for the $\mathbb{Z}_p$-extension L_∞/L where $L_\infty = LF_{\mathrm{nc}}$. Now,

$$\begin{aligned} s\left(Y(A/L_\infty)\right)p^n &= r_p\left(R_{p^\infty}\left(A/L_n\right)\right) + O(1) \\ &\geq r_p\left(A\left(L_n\right)[p]\right)\left(s\left(X(L_\infty)\right)p^n\right) - 4ds_0p^n + O(1) \\ &\geq r_p\left(A\left(L_n\right)[p]\right)\left(t - p(p-1) - 4ds_0\right)p^n + O(1). \end{aligned}$$

The first equality follows from Lemma 3.11. The last line follows from Proposition 4.5 by observing that at least tp^n primes of F_n ramify in L_n (by construction) and $[L_n : \mathbb{Q}] = p(p-1)p^n$. By *Step 2* we know that t can be chosen to be arbitrarily large. Therefore, given $N \geq 1$ there exists L/F such that

$$s\left(Y(A/L_\infty)\right) \geq N.$$

Since $\mu\left(Y(A/L_\infty)\right) \geq s\left(Y(A/L_\infty)\right)$, the theorem follows.

Remark 4.7. The condition A/F is an Abelian variety such that $A(F)[p] = 0$ is a mild condition. We can base change to a finite extension F'/F such that $A(F')[p] \neq 0$. Theorem 4.2 can then be stated (and proved) in terms of F' provided the (universal) analogue of the Weak Leopoldt Conjecture holds.

5. Growth of Fine Selmer Groups in Infinite Class Field Towers

In this section, we consider growth of fine Selmer groups in non-Abelian and non-p adic analytic (infinite) extensions. Let K be an imaginary quadratic field and $\mathfrak{p}$ be an unramified prime in K. Further suppose $p \geq 5$. Let F/K be any finite Galois extension where $\mathfrak{p}$ is unramified and E/F be an elliptic curve with CM by $\mathcal{O}_K$. Let S be a finite set of primes in F containing the Archimedean primes, primes above p, and the primes of bad reduction of E.

Definition 5.1. *Set $H_S(F)$ to be the p-Hilbert S-class field of F. It is the maximal Abelian unramified p-extension of F such that all the primes in S split completely.*

By class field theory, $\mathrm{Gal}\left(H_S(F)/F\right) \simeq \mathrm{Cl}_S(F)$, the S-ideal class group of F. Set F_∞^S to be the maximal unramified p-extension of F such that all the primes in S split completely. Set $\Gamma = \Gamma_F = \mathrm{Gal}(F_\infty^S/F)$ and write $\{\Gamma_n\}_{n\geq 0}$ for its derived series. For each $n \geq 0$, the fixed field F_n corresponding to Γ_n is the p-Hilbert S-class field of F_{n-1}.

Set $r_1(F)$ (resp. $r_2(F)$) to mean the number of real (resp. number of pairs of complex) places of F. In [8], Golod and Shafarevich showed that if

$$r_p\left(\mathrm{Cl}_S(F)\right) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + \left|S \setminus S_\infty\right|}$$

then Γ_F is infinite. Stark posed the following natural question.

Question. *Does $r_p\left(\mathrm{Cl}_S\left(F_n\right)\right)$ tend to infinity as n tends to infinity?*

For any pro-p group G, $\mathbb{Z}/p\mathbb{Z}$ is a G-module with trivial G-action. In particular, take $G = \Gamma_n$, the n-th term of the derived series. By class field theory and finiteness of $\mathrm{Cl}(F_n)$ (and hence of $\mathrm{Cl}_S(F_n)$) we know

$$\begin{aligned}
r_p\left(\mathrm{Cl}_S(F_n)\right) &= r_p\left(\mathrm{Cl}_S(F_n)/p\right)\\
&= \dim_{\mathbb{Z}/p\mathbb{Z}}\left(H^1\left(\mathrm{Gal}(H_S(F_n)/F_n),\ \mathbb{Z}/p\mathbb{Z}\right)\right)\\
&= \dim_{\mathbb{Z}/p\mathbb{Z}}\left(H^1\left(\Gamma_n, \mathbb{Z}/p\mathbb{Z}\right)\right)\\
&= \dim_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Hom}\left(\Gamma_n, \mathbb{Z}/p\mathbb{Z}\right).
\end{aligned}$$

This is the number of minimal generators of Γ_n. By a result of Lubotzky and Mann [18, Theorem A], $r_p\left(\mathrm{Cl}_S\left(F_n\right)\right)$ tends to infinity if and only if Γ_F is not p-adic analytic. It follows that the question posed by Stark is equivalent to asking whether Γ_F is p-adic analytic. The following result was shown independently by Boston and Hajir [3], [9]. A different proof was also provided by Matar [19].

Theorem 5.2. *Let F be a number field. If either of the following inequalities hold*

$$\begin{aligned}
r_p\left(\mathrm{Cl}(F)\right) &\geq 2 + 2\sqrt{r_1(F) + r_2(F)},\\
\textit{or } r_p\left(\mathrm{Cl}_S(F)\right) &\geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + \left|S \setminus S_\infty\right|},
\end{aligned}$$

then Γ_F is not p-adic analytic.

The main result we prove in this section is the following.

Theorem 5.3. *Let F be a number field containing the imaginary quadratic field K. Let $p \neq 2,\ 3$ be a prime that splits in K as $\mathfrak{p}\bar{\mathfrak{p}}$ such that $\mathfrak{p}$ is unramified in F/K. Suppose $E(F)[\mathfrak{p}] \neq 0$. Let S be a finite set of primes in F containing the Archimedean primes, primes above $\mathfrak{p}$ and primes where E has bad reduction. Assume that F satisfies the Golod-Shafarevich inequality*

$$r_p\left(\mathrm{Cl}_S(F)\right) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + \left|S \setminus S_\infty\right|}.$$

Let F_∞^S be the maximal unramified (non-constant) pro-p extension of F where primes in S split completely and let F_n be the n-th layer of the p-Hilbert S-class field tower. Then the p-rank of the $\mathfrak{p}$-fine Selmer group of E/F_n becomes arbitrarily large as $n \to \infty$.

Proof. In Lemma 3.5, we proved that if $E(F)[p] \neq 0$,

$$r_p\left(R_p^S\left(E/F\right)\right) \geq r_p\left(\mathrm{Cl}_S\left(F\right)\right) r_p\left(E\left(F\right)[p]\right) - 2.$$

In the exact same way, it is possible to prove the following inequality for every extension F_n/F,

$$r_p\left(R_{\mathfrak{p}}^S\left(E/F_n\right)\right) \geq r_p\left(\mathrm{Cl}_S\left(F_n\right)\right) r_p\left(E\left(F_n\right)[\mathfrak{p}]\right) - 2. \tag{8}$$

Since we assume that F satisfies the Golod-Shafarevich inequality, by Theorem 5.2 we know Γ_F is not p-adic analytic. By the theorem of Lubotzky-Mann, $r_p\left(\mathrm{Cl}_S\left(F_n\right)\right)$ tends to infinity as $n \to \infty$. The theorem now follows from Inequality 8. □

Remark 5.4.
1. Theorem 5.3 is true for any d-dimensional CM Abelian variety.
2. Theorem 5.3 is the fine Selmer variant of [24, Theorem 4] and implies [17, Theorem 7.2].

The author thanks the referee for pointing out that the following theorem can be proved. It is the class group variant of the above theorem.

Let p be a fixed odd prime and let F be a number field. Let F_∞ be the maximal unramified p-extension of F. Now, set $\Sigma = \Sigma_F = \mathrm{Gal}(F_\infty/F)$. Let $\{\Sigma_n\}_{n\geq 0}$ be the derived series of Σ. For every $n \geq$, the fixed field F_{n+1} corresponding to Σ_{n+1} is the p-Hilbert class field of F_n.

By class field theory and finiteness of $\mathrm{Cl}(F_n)$, we know

$$r_p\left(\mathrm{Cl}(F_n)\right) = \dim_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Hom}\left(\Sigma_n, \mathbb{Z}/p\mathbb{Z}\right).$$

This is the number of minimal generators of Σ_n. By a result of Lubotzky and Mann [18, Theorem A], the p-rank $r_p\left(\mathrm{Cl}\left(F_n\right)\right)$ tends to infinity if and only if Σ_F is not p-adic analytic.

Theorem 5.5. *Let K be an imaginary quadratic field and let F be a finite Galois extension of K. Let $p \neq 2,\ 3$ be a prime that splits in K as $\mathfrak{p}\bar{\mathfrak{p}}$. Let E/F be an elliptic curve with CM by $\mathcal{O}_K$ and suppose $E(F)[p] \neq 0$. Assume that F satisfies the inequality*

$$r_p\left(\mathrm{Cl}(F)\right) \geq 2 + 2\sqrt{r_1(F) + r_2(F)}. \tag{9}$$

Let F_∞ be the maximal unramified p-extension of F and let F_n be the n-th layer of the p-Hilbert class field tower. Then the p-rank of the p^∞-fine Selmer group of E/F_n becomes arbitrarily large as $n \to \infty$.

Proof. Let $L = F(E[p])$. Since E is an elliptic curve with CM, it has good reduction at all the primes in L. Denote by H_F, the p-Hilbert class field of F and set $L' = LH_F$. Notice that by hypothesis, the group $G = \mathrm{Gal}(L'/L) \simeq \mathrm{Gal}(H_F/F)$ has order coprime to p.

To prove the theorem, it suffices to show the following injection

$$H^1\left(\mathrm{Gal}\left(H_F/F\right),\ E(H_F)[p^\infty]\right) \hookrightarrow R_{p^\infty}(E/F). \tag{10}$$

Indeed, once this injection is established, one has

$$r_p\left(R_{p^\infty}\left(E/F\right)\right) \geq r_p\left(H^1\left(\mathrm{Gal}\left(H_F/F\right),\ E(H_F)[p^\infty]\right)\right) \tag{11}$$

$$\geq r_p\left(\mathrm{Cl}(F)\right) r_p\left(E\left(F\right)[p]\right) - 2 \tag{12}$$

where the second inequality can be proven in the same way as Lemma 3.5 upon observing that $r_p\left(E\left(F\right)\left[p\right]\right) = r_p\left(E\left(F\right)\left[p^\infty\right]\right)$. Since F satisfies Inequality 9, by Theorem 5.2 we know that Σ_F is not p-adic analytic. By the theorem of Lubotzky-Mann, the p-rank $r_p\left(\mathrm{Cl}\left(F_n\right)\right)$ tends to infinity as $n \to \infty$. The theorem now follows From Inequality 12.

It remains to show that Injection 10 holds. By an argument used in proving Lemma 3.5, we have an injection

$$H^1\left(G,\ E\left(L'\right)\left[p^\infty\right]\right) \hookrightarrow R_{p^\infty}\left(E/L\right).$$

Taking Gal(L/F)-invariant, it follows that

$$H^1\left(G,\ E\left(L'\right)\left[p^\infty\right]\right)^{\mathrm{Gal}(L/F)} \hookrightarrow R_{p^\infty}\left(E/L\right)^{\mathrm{Gal}(L/F)}.$$

Injection 10 follows from the above injection by the same argument as in [16, Theorem 4.1(ii)]. This finishes the proof of the theorem. □

Remark 5.6. (1) In general, this theorem can be proven for d-dimensional CM Abelian varieties. For the case of CM elliptic curves, the theorem holds for $p = 3$, provided K is not $\mathbb{Q}(\sqrt{-3})$ [16, Remark 4.5].

(2) Using analogous arguments, it should be possible to prove under the same hypothesis that the p-rank of the p-fine Selmer group of E/F_n becomes arbitrarily large as $n \to \infty$.

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