What is ... an Euler System?

A non-math defn: An ES is a coll^n of "geometric objects" (pt on elliptic curves, units of nf, cycles in Chow groups etc) which satisfy 2 basic features

(i) can be connected to L-funcns

(ii) can be made to vary in p-adic and tame families

[Reference: p-adic L-functions and Euler Systems ... A tale in two trilogies]

(1) Circular Units
(2) Elliptic Units
(3) Heegner Points

} Studied in the late 80's

more recent (back in ~2010)

(1') Beilinson - Kato elts
(2') Beilinson - Flach
(3') Diagonal cycles in triple products of modular curves

} Garrett-Rankin-Selberg Euler Systems

We will hear new analogues of 1', 2', 3' next week! (Sarah-David)

(1) GSp_4 (2) $GSp_4 \times Gl_2$ (3) $GSp_4 \times Gl_2 \times Gl_2$

1.1 Cyclotomic Units (developed by Kummer and rediscovered by Thaine)

defn: A circular unit is an elt of the form
$$\begin{cases} 1-\zeta_n & (n \neq \text{prime}) \\ \dfrac{1-\zeta_n^a}{1-\zeta_n^b} & (n = \text{prime}) \end{cases}$$

They belong to $\mathcal{O}_{F_n}^\times$ where $F_n = \mathbb{Q}(\zeta_n)$

Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$

st $\chi(-1) = 1$ (even char)

$$U(\chi) = \prod_{a \in (\mathbb{Z}/N)^\times} (1 - \zeta_n^a)^{\chi^{-1}(a)} \in \left(\mathcal{O}_{F_n}^\times \otimes \mathbb{Z}[\chi]\right)^\chi$$

$$L'(0,\chi) \sim L(1,\chi) \sim \log|U(\chi)|$$

Dirichet L-funcn

There is a p-adic analogue (Kubota - Leopoldt)

$$L_p(1,\chi) \sim \log_p(U(\chi))$$

p-adic logarithm

Eisenstein series of wt $k \geq 2$ (even)

$$E_{k,\chi}(q) = L(1-k,\chi) + 2\sum_{n\geq 1} \sigma_{k-1,\chi}(n) q^n$$

where $\sigma_{k-1,\chi}(n) = \sum_{d|n} \chi(d) d^{k-1}$

modify to get

$$E_{k,\chi}^{(p)}(q) = L_p(1-k,\chi) + 2\sum_{n\geq 1} \sigma_{k-1,\chi}^{(p)}(n) q^n$$

where $\sigma_{k-1,\chi}^{(p)} = \sum_{\substack{p \nmid d \\ d|n}} \chi(d) d^{k-1}$

By p-adic interpolation

$$E_{0,\chi}^{(p)} = L_p(1,\chi) + 2\sum_{n\geq 1}\Big(\sum_{\substack{p\nmid d \\ d|n}} \chi(d) d^{-1}\Big) q^n$$

This is a p-adic modular form

§ Siegel Units

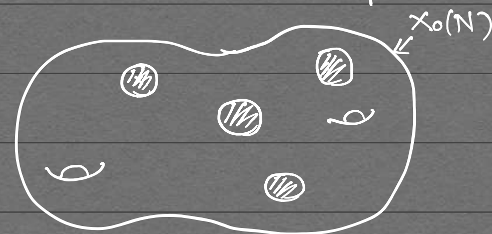
$1 \leq a \leq N$

$$g_a(q) = q^{1/k}(1-\zeta^a)\prod_{n>0}(1-q^n\zeta_N^a)(1-q^n\zeta_N^{-a})$$

on $X_1(N)$

needs to be modified to get an honest unit.

Φ = canonical lift of Frobenius on $X_0(N)^{ord}$ $(p \nmid N)$



$$g_a^{(p)} = \Phi^*(g_a) g_a^{-p}$$
$$= g_{pq}(q^p) g_a(q)^p$$

$\Phi(A) = A/\sigma_{can}$

The logarithm of $g_a^{(p)}$ is analytic on $X_1(N)^{ord}$

If we write

$$\sum_{a \in (\mathbb{Z}/N)^\times} \chi(a) \log g_a^{(p)} = h_\chi^{(p)} = \log_p(U(\chi))\ 1 - \chi(p)p^{-1} / g_a(\chi^{-1})$$
$$+ 2 \sum_{n \geq 1} \Big(\sum_{p \nmid d \mid n} \chi(d) d^{-1} \Big) q^n$$

$$g_a^{(p)} : X_1(N)^{ord}(\mathbb{Q}) \longrightarrow \{ z \in \mathbb{C}_p,\ |z-1| < 1 \}$$

$\Rightarrow$ Leopoldt's formula

Theme of p-adic Variation:

$U(\chi)$ encodes a p-adic limit of classical special values

View it as:

Classical special values encode info about p-adic limits of circular units.

$U(\chi)$ comes in norm compatible fam over the cyclotomic $\mathbb{Z}_p$-ext^n

$$U_{\chi,n} = \prod_a^N \left(1 - \zeta_{Np^n}^a\right)^{\chi^{-1}(a)} \in \left(\mathcal{O}_{F_{Np^n}}^\times \otimes \mathbb{Z}[\chi]\right)^\chi$$

$$\mathrm{norm}_{F_{Np^n}}^{F_{Np^{n+1}}} U_{\chi,n+1} = U_{\chi,n} \quad (n \geq 1)$$

Consider $\delta : \mathcal{O}_F^\times \otimes \mathbb{Z}_p \longrightarrow H^1(F, \mathbb{Z}_p(1))$

$u \longmapsto \left(\sigma \mapsto \left(\sigma u^{1/p^n} / u^{1/p^n}\right)_{n \geq 1}\right)$

kappa

$\underline{\kappa}_{\chi,n} = \delta(U_{\chi,n}) \in H^1(F_{Np^n}, \mathbb{Z}_p(1))$

χ (fixed) takes values in $\mathbb{Z}_p$ (for not^n simplicity)

$(\underline{\kappa}_{\chi,n})_{n\geq 1} \in \varprojlim_n H^1(F_{p^n}, \underbrace{\mathbb{Z}_p(1)}(\chi))$

$\underline{\kappa}_{\chi,\infty}$

acts via cyclotomic char + fin ord char.

Shapiro's lemma

$\varprojlim_n H^1(\mathbb{Q}, \underbrace{\mathbb{Z}_p[G_n]}(1)(\chi))$

$G_n = \mathrm{Gal}(F_n/\mathbb{Q})$

$G_{\mathbb{Q}}$-module, $\sigma.(\lambda) = \chi(\sigma)\chi_{cyc}(\sigma)(\sigma\lambda)$

$H^1(\mathbb{Q}, \Lambda_{cyc}(1)(\chi))$

(Iwasawa cohomology)

$\Lambda_{cyc}(\chi)$ is a Λ-adic rep^n which interpolates all the Tate twists $\mathbb{Z}_p(k)(\chi)$, $\mathbb{Z}_p(k)(\chi \cdot \chi_{p^n})$

$\underline{\kappa}_{\chi,\infty} \in H^1(\mathbb{Q}, \Lambda_{cyc}(\chi))$ gives a coll^n of classes $\underline{\kappa}_\chi(k, \chi_{p^n})$

$\in H^1(\mathbb{Q}, \mathbb{Z}_p(k)(\chi \cdot \chi_{p^n}))$

Leopoldt $\Rightarrow$ $\underline{\kappa}_\chi(1, \chi_n) \longleftrightarrow L_p(1, \chi\chi_n)$

expect a relation

$$\underline{\kappa}_\chi(k, 1) \longleftrightarrow L_p(k, \chi)$$

Perrin-Riou map

$$H^1(\mathbb{Q}, \Lambda_{cyc}(\chi)) \longrightarrow \widetilde{\Lambda}$$
$$\underline{\kappa}_{\chi,\infty} \longmapsto L_p(\chi, s)$$

The arithmetic application : related to the Bloch-Kato conj

$L(V,0) \leftrightarrow$ controls the size of $H^1_f(\mathbb{Q},V)$

BK conj $\qquad L(V,0) \neq 0 \iff H^1_f(\mathbb{Q},V)$ is fin

$\Rightarrow$ what one gets normally

Iwasawa Main Conjecture
(Thaine/Rubin late 80's)

1.2 Elliptic Units : The value of a Siegel unit at a CM point of modular curve

Class number formula replaced by Kronecker Limit formula

$$\begin{array}{c} H \\ \Big| \chi \\ K \end{array} \quad \text{Ab} \qquad u_H , \quad u(\chi) \in (\mathcal{O}_H^\times)^\chi$$

KLF : $\log |U(\chi)| \longleftrightarrow L'(K, \chi, 0)$

K Imag quad

To get a p-adic L-funcns one needs to do more work
(Katz L-funcn)

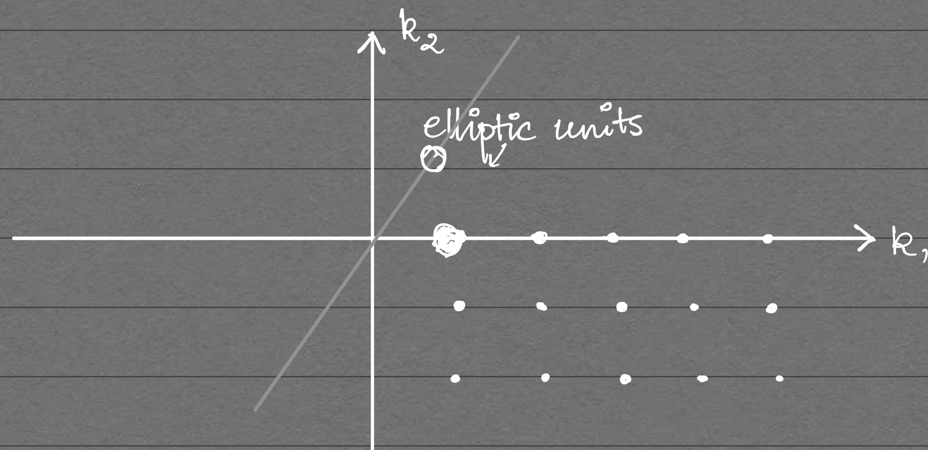
A char $\psi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ is said to have ∞-type (k_1, k_2)
if $\psi(z) = z^{k_1} \bar{z}^{k_2} \quad \forall z \in (K \otimes \mathbb{R})^\times$

ψ can also be thought as a funcn $\psi : I(K)^\times \to \mathbb{C}^\times$
(Hecke char)

$$a \mapsto a^{k_1} \bar{a}^{-k_2}$$

$$\forall a \equiv 1 \pmod{m}$$

If $k_1 > 0$, $k_2 \leq 0$ & ψ is of ∞-type (k_1, k_2) then
$L(\psi, 0)$ is critical



$\log U(x) \leftrightarrow L_p(x, 1)$ Coates-Yager
Coates-Wiles

Deuring + Hasse $\Rightarrow$ E w/ CM by K
$L(E,s) \longleftrightarrow L(\psi, s)$
$\psi = (1, 0)$

Coates-Wiles: $h(K) = 1$
$L(E,1) \neq 0 \Rightarrow E(\mathbb{Q}) < \infty$

$$\underline{\kappa}(\psi) \in H^1(K, T_pE)$$
$$\uparrow \delta$$
$$E(K) \otimes \mathbb{Z}_p$$

$\underline{\kappa}(\psi) \in \delta(E(K_v)) \ \forall v \neq p$

So, im of $\underline{\kappa}(\psi)$ in $\dfrac{H^1(K_p, T_pE)}{\delta(E(K_p))}$ is non-zero $\Leftrightarrow$ $L(\psi, 0) \neq 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad L(E,1)$

$\dim_{\mathbb{Q}_p} (E(\mathbb{Q}_p) \otimes \mathbb{Q}_p) = 1$, $\dim_{\mathbb{Q}_p} (H^1(\mathbb{Q}_p, T_pE \otimes \mathbb{Q}_p)) = 2$

There is a duality

$$H^1(\mathbb{Q}_p, T_pE) \times H^1(\mathbb{Q}_p, T_pE) \to H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) = \mathbb{Z}_p$$

Claim: The submodule

$\delta(E(\mathbb{Q}_p))$ is its own annihilator under this duality

$$E(\mathbb{Q}_p) \otimes \mathbb{Q}_p \times \frac{H^1(\mathbb{Q}_p, T_pE)}{\delta(\quad)} \longrightarrow \mathbb{Q}_p$$

perfect

$\mathrm{res}_p(\underline{\kappa}(\psi))$ is orthogonal to the image of $E(\mathbb{Q})$ in $E(\mathbb{Q}_p) \otimes \mathbb{Q}_p$

If $\mathrm{Im}(E(\mathbb{Q}))$ in $E(\mathbb{Q}_p)$ is fin

$\Rightarrow E(\mathbb{Q})$ is fin