HIGHER COLEMAN THEORY (VERSION 18/11/20)

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Abstract. We develop local cohomology techniques to study the finite slope part of the cohomology of Shimura varieties. The local cohomology groups we consider are defined by using a stratification on the Shimura variety obtained from the Bruhat stratification on a Flag variety via the Hodge-Tate period map. Overconvergent modular forms are a particular case of these local cohomologies. We construct a spectral sequence from local cohomology to cohomology. We are able to obtain vanishing theorems for the cohomology, as well as classicality theorem comparing local and classical cohomology. We also develop eigenvarieties by p-adic deformation of the local cohomology groups. As an application, we prove some new properties of Galois representations arising from certain non-regular algebraic cuspidal automorphic forms.

Contents

1. Introduction

Let G be a split reductive group over a field k. Let B be a Borel subgroup, containing a maximal torus T. We let $X^\star(T)$ be the group of characters of T. Let $FL = B\backslash G$ be the Flag variety for G and $\pi : G \to FL$ be the projection map. Let d be the dimension of FL. Let W be the Weyl group of G and ρ be half the sum of the positive roots. We have a length function $\ell : W \to [0, d]$. Let $w_0 \in W$ be the longest element. For any $\kappa \in X^\star(T)$, we define a G-equivariant line bundle $\mathcal{L}_\kappa$ over FL. Its sections over an open subset $U \hookrightarrow FL$ are functions $f : \pi^{-1}(U) \to \mathbb{A}^1$ such that $f(bu) = w_0\kappa(b)f(u)$ for all $u \in \pi^{-1}(U)$. The right action of G on FL given by $(Bg)g' = Bgg'$ induces a left action on the cohomology groups $\mathrm{H}^i(FL, \mathcal{L}_\kappa)$. If κ is dominant, then $\mathrm{H}^0(FL, \mathcal{L}_\kappa)$ is a highest weight representation of weight κ. We introduce the dotted action of W on $X^\star(T)$: $w \cdot \kappa = w(\kappa + \rho) - \rho$.

The following classical Borel-Weil-Bott theorem describes the cohomology of the sheaves $\mathcal{L}_\kappa$ over FL when the characteristic of k is 0:

Theorem 1.1 ([Jan03], 5.5, corollary)**.** *Assume that* $\mathrm{char}(k) = 0$*. Let* $\kappa \in X^\star(T)$ *then:*

1. *If there exists no* $w \in W$ *such that* $w \cdot \kappa$ *is dominant then* $\mathrm{H}^i(FL, \mathcal{L}_\kappa) = 0$ *for all* i*.*
2. *If there exists* $w \in W$ *such that* $w \cdot \kappa$ *is dominant, then there is a unique such* w*, and* $\mathrm{H}^i(FL, \mathcal{L}_\kappa) = 0$ *if* $\ell(w) \neq i$*, while* $\mathrm{H}^{\ell(w)}(FL, \mathcal{L}_\kappa)$ *is the highest weight* $w \cdot \kappa$ *representation.*

Following [Kem78], section 12, one can study the cohomology of the sheaves $\mathcal{L}_\kappa$ over FL with the help of the Bruhat stratification $FL = \cup_{w\in W} B\backslash BwB$ and build a Cousin complex which computes the cohomology. Namely, for all $w \in W$, let X_w be the Schubert variety equal to the closure of $B\backslash BwB$ in FL. Consider the stratification of FL by closed subsets $FL = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_d \supseteq Z_{d+1} = \emptyset$ where $Z_i = \cup_{w,\ell(w)=d-i} X_w$. Then there is a Grothendieck-Cousin complex $\mathcal{C}ous(\kappa)$

$$0 \to \mathrm{H}^0_{Z_0/Z_1}(FL, \mathcal{L}_\kappa) \to \mathrm{H}^1_{Z_1/Z_2}(FL, \mathcal{L}_\kappa) \to \cdots \to \mathrm{H}^d_{Z_d/Z_1}(FL, \mathcal{L}_\kappa) \to 0$$

which computes $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)$. The cohomologies $\mathrm{H}^i_{Z_i/Z_{i+1}}(FL, \mathcal{L}_\kappa)$ are by definition certain cohomology groups with support on the Bruhat cells of codimension i.

The modules appearing in the Cousin complex are infinite dimensional, but the action of the torus is very easy to determine and one can prove the following result which is valid in all characteristics.

Proposition 1.2 (Proposition 3.7)**.** *Let $\kappa \in X^\star(T)$ and let $C(\kappa) = \{w \in W,\ w(\kappa+\rho) \in X^\star(T)^+\}$. Let $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)^{bw}$ be the big weight part of $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)$, which is the direct factor where the weights of T are $> w\cdot\kappa$ for all $w \notin C(\kappa)$. Then the cohomology $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)^{bw}$ is a prefect complex of amplitude* $[\min_{w\in C(\kappa)} \ell(w), \max_{w\in C(\kappa)} \ell(w)]$.

One can show that the Cousin complex is a complex in the BGG category $\mathcal{O}$. In characteristic 0, we derive a full proof of theorem 1.1 as a combination of some basic properties of the category $\mathcal{O}$ and the description of the action of the torus on the Cousin complex. See theorem 3.9.

The main theme of this paper is the coherent cohomology of Shimura varieties. The ideas we will employ use the close relation between Shimura varieties and flag varieties, as provided by the Hodge-Tate period map constructed in [Sch15] and refined in [CS17]. We develop methods from local cohomology similar to the Grothendieck-Cousin complex of [Kem78]. Let (G, X) be a Shimura datum. By definition G is a reductive group over $\mathbb{Q}$ and X is a complex analytic space equipped with an action of $G(\mathbb{R})$ and satisfying a list of axioms ([Del79]). There are two opposite parabolic subgroups of G attached to (G, X), called P_μ and P^{std}_μ. The space X embeds $G(\mathbb{R})$-equivariantly as an open subspace of $FL^{std}_{G,\mu}(\mathbb{C}) = G/P^{std}_\mu(\mathbb{C})$. This is the Borel embedding.

For any neat compact open subgroup $K \subseteq G(\mathbb{A}_f)$, we let $S_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ be the corresponding Shimura variety over $\mathbb{C}$. This is a finite disjoint union of arithmetic quotients of X.

Any representation of the Levi M_μ of P^{std}_μ defines a G-equivariant vector bundle over $FL^{std}_{G,\mu}$. Let $Z_s \subseteq G$ be the maximal torus of the center of G which is not split but splits over $\mathbb{R}$. We let $M^c_\mu = M_\mu/Z_s$. By pull back to X and descent to $S_K(\mathbb{C})$, we obtain a functor from the category of representations of M^c_μ to the category of vector bundles on $S_K(\mathbb{C})$, whose essential image is the set of (totally decomposed) automorphic vector bundles.We make a choice of Borel subgroup contained in P_μ, and we let T be a maximal torus contained in this Borel and let $T^c = T/Z_s$. We label irreducible representations of M^c_μ by their highest weight in $X_\star(T^c)^{M_\mu,+}$. For any $\kappa \in X_\star(T^c)^{M_\mu,+}$ we let $\mathcal{V}_\kappa$ be the corresponding vector bundle over $S_K(\mathbb{C})$.

The Shimura variety $S_K(\mathbb{C})$ has a structure of algebraic variety S_K defined over a number field E, called the reflex field. For a combinatorial choice Σ of cone decomposition, there are algebraic compactifications $S^{tor}_{K,\Sigma}$ whose boundary $D_{K,\Sigma} =$

$S^{tor}_{K,\Sigma} \setminus S_K$ is a Cartier divisor. The vector bundles $\mathcal{V}_\kappa$ admit models over S_K and canonical extensions $\mathcal{V}_{\kappa,\Sigma}$ to $S^{tor}_{K,\Sigma}$. We denote by $\mathcal{V}_{\kappa,\Sigma}(-D_{K,\Sigma})$ the sub-canonical extension ([Mil90], [Har90a]).

This paper is devoted to the study of the cohomologies of weight κ (which are independent of Σ): $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma})$, $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}(-D_{K,\Sigma}))$ as well as the interior cohomology:

$$\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}) = \mathrm{Im}(\mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}(-D_{K,\Sigma})) \to \mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma})).$$

The most fundamental example of a Shimura datum is the Siegel datum $(\mathrm{GSp}_{2g}, \mathcal{H}_g)$ for $g \in \mathbb{Z}_{\geq 1}$, where $\mathcal{H}_g$ is the Siegel space. The corresponding Shimura variety S_K is a a moduli space of abelian schemes of dimension g with a polarization and a level structure prescribed by K. We assume in this work that the datum (G, X) is an abelian Shimura datum, therefore S_K is (closely related to) a moduli space of abelian varieties with certain extra structures (endomorphism, polarization, level structure, Hodge tensors...). For the rest of this paper, we fix a prime p and we also fix an embedding of $E \hookrightarrow \overline{\mathbb{Q}}_p$. We assume that $G_{\mathbb{Q}_p}$ is quasi-split.

In summary, our two assumptions are:

Assumption 1.3. The Shimura datum (G, X) is abelian and $G_{\mathbb{Q}_p}$ is quasi-split.

We fix a compact open subgroup $K^p \subseteq G(\mathbb{A}^p_f)$. We now consider the following $G(\mathbb{Q}_p)$-representations arising from the cohomology of Shimura varieties:

$$\mathrm{H}^i(K^p, \kappa) = \mathrm{colim}_{K_p} \mathrm{H}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_{\kappa,\Sigma}),$$

$$\mathrm{H}^i(K^p, \kappa, cusp) = \mathrm{colim}_{K_p} \mathrm{H}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_{\kappa,\Sigma}(-D_{K^pK_p,\Sigma}))$$

as well as $\overline{\mathrm{H}}^i(K_p, \kappa) = \mathrm{Im}(\mathrm{H}^i(K^p, \kappa, cusp) \to \mathrm{H}^i(K^p, \kappa))$.

We define a first direct summand as $G(\mathbb{Q}_p)$-representation

$$\mathrm{H}^i(K^p, \kappa)^{fs} \subseteq \mathrm{H}^i(K^p, \kappa)$$

that we call the finite slope part of $\mathrm{H}^i(K^p, \kappa)$. It contains all the irreducible smooth $G(\mathbb{Q}_p)$-subquotients which can be embedded in a principal series representation $\iota^{G(\mathbb{Q}_p)}_{B(\mathbb{Q}_p)}\lambda$ for a character λ of $T(\mathbb{Q}_p)$. We define direct summands

$$\mathrm{H}^i(K^p, \kappa)^{sss^M(\kappa)} \subseteq \mathrm{H}^i(K^p, \kappa)^{ss^M(\kappa)} \subseteq \mathrm{H}^i(K^p, \kappa)^{fs}$$

that we call the strongly small slope and small slope part of $\mathrm{H}^i(K^p, \kappa)$. These are direct factors which contains all irreducible subquotient smooth $G(\mathbb{Q}_p)$-representations of $\mathrm{H}^i(K^p, \kappa)^{fs}$ which can be embedded in a principal series representation $\iota^{G(\mathbb{Q}_p)}_{B(\mathbb{Q}_p)}\lambda$ for a character λ of $T(\mathbb{Q}_p)$ whose p-adic valuation is small with respect to κ, in a sense that is made precise in the paper (section 5.11).

We adopt similar definitions for the cuspidal and interior cohomology.

Example 1.4. In the case of modular curves and the sheaf of weight k modular forms, an eigenvalue α for the U_p-operator acting on the cohomology has small valuation if $v(\alpha) < k - 1$ when $k \geq 2$, $v(\alpha) < 1 - k$ when $k \leq 0$, while there is no condition if $k = 1$.

For any $\kappa \in X_\star(T^c)^{M_\mu,+}$, there is a range $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)] \subseteq [0, d]$ where $d = \dim S_K$ where we expect to see interesting (for example tempered) cohomology classes in the cohomology of that given weight. Here is a combinatorial description of this range. Let $C(\kappa)^+ = \{w \in W, w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^-_{\mathbb{Q}}\}$. Put $\ell_{\min}(\kappa) = \inf_{w\in C(\kappa)^+}\ell(w)$, $\ell_{\max}(\kappa) = \sup_{w \in C(\kappa)^+}\ell(w)$. Our first theorem is the following:

Theorem 1.5 (Theorem 5.82 and 6.49)**.** *For any $\kappa \in X_\star(T^c)^{M_\mu,+}$,*

(1) $\overline{\mathrm{H}}^i(K^p,\kappa)^{ss^M(\kappa)}$ *is concentrated in the range* $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$,
(2) $\mathrm{H}^i(K^p,\kappa, cusp)^{sss^M(\kappa)}$ *is concentrated in the range* $[0, \ell_{\max}(\kappa)]$,
(3) $\mathrm{H}^i(K^p,\kappa)^{sss^M(\kappa)}$ *is concentrated in the range* $[\ell_{\min}(\kappa), d]$.

Remark 1.6. We believe that our assumption $ss^M(\kappa)$ is optimal in order to obtain such a result. We note that all ordinary class occuring in the cohomology will satisfy $ss^M(\kappa)$. We conjecture that point (2) and (3) of the theorem should also hold under the weaker small slope assumption. If $\kappa+\rho$ is sufficiently far away from the walls that do not contain it, any ordinary class will satisfy $sss^M(\kappa)$.

Remark 1.7. If the weight κ is such that $\kappa+\rho$ is regular, then $C(\kappa)^+$ consists of a single element. The interior cohomology is therefore concentrated in a single degree $\ell_{\min}(\kappa) = \ell_{\max}(\kappa)$.

Remark 1.8. In [Lan16] or [BHR94], an analog of this theorem is proved without any small slope condition, but with a regularity condition on the weight κ.

Remark 1.9. There is a classical Archimedean result due to the combined works of Blasius-Harris-Ramakrishnan, Mirkovich, Schmid and Williams (see [Har90a], theorems 3.4 and 3.5) which asserts that for any compact open $K \subseteq G(\mathbb{A}_f)$, the tempered at infinity interior cohomology $\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma})^{temp}$ is concentrated in the range $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$. This temperedness condition is a growth condition on the harmonic functions on X which represents the cohomology classes. The cohomology $\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma})$ is entirely expressed using automorphic forms π on G, and this temperedness condition is therefore a condition on π_∞. On the other hand, the small slope condition is a condition on the p-adic "size" of the coefficients of the local representation π_p of the group $G(\mathbb{Q}_p)$. We therefore think of the small slope condition as a p-adic version of temperedness.

Let $\nu \in X^\star(T^c)^+$ and let W_ν be the corresponding irreducible representation of G and $W_\nu^\vee$ its contragrediant. We can attach to it a local system $\mathcal{W}_\nu^\vee$ on $S_K(\mathbb{C})$. We have the Betti cohomology groups $\mathrm{H}^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$, $\mathrm{H}^\star_c(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$ and the interior cohomology $\overline{\mathrm{H}}^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee) = \mathrm{Im}(\mathrm{H}^\star_c(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee) \to \mathrm{H}^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee))$. We can consider

$$\mathrm{H}^i(K^p, \mathcal{W}_\nu^\vee) = \mathrm{colim}_{K_p}\mathrm{H}^i(S_{K^pK_p}(\mathbb{C}), \mathcal{W}_\nu^\vee)$$

,

$$\mathrm{H}^i_c(K^p, \mathcal{W}_\nu^\vee) = \mathrm{colim}_{K_p}\mathrm{H}^i_c(S_{K^pK_p}(\mathbb{C}), \mathcal{W}_\nu^\vee)$$

as well as $\overline{\mathrm{H}}^i(K_p, \mathcal{W}_\nu^\vee) = \mathrm{Im}(\mathrm{H}^i_c(K^p, \mathcal{W}_\nu^\vee) \to \mathrm{H}^i(K^p, \mathcal{W}_\nu^\vee))$. One can consider the finite slope part (denoted by a supscript fs), the small slope part (denoted by a supscript $ss_b(\nu)$) and strongly small slope part (denoted by a supscript $sss_b(\nu)$) of these cohomology.

Using Faltings's BGG spectral sequence we deduce easily:

Theorem 1.10 (Theorem 5.82 and 6.49)**.** *For any $\nu \in X_\star(T^c)^+$,*

(1) $\overline{\mathrm{H}}^i(K_p, \mathcal{W}_\nu^\vee)^{ss_b(\nu)}$ *is concentrated in the middle degree d,*

(2) $\mathrm{H}^i_c(K^p, \mathcal{W}^\vee_\nu)^{sss_b(\nu)}$ *is concentrated in the range* $[0, d]$,
(3) $\mathrm{H}^i(K^p, \mathcal{W}^\vee_\nu)^{sss_b(\nu)}$ *is concentrated in the range* $[d, 2d]$.

Remark 1.11. The conditions $ss_b(\nu)$ and $sss_b(\nu)$ are the union of all conditions $ss^M(\kappa)$ and $sss^M(\kappa)$ respectively where $\kappa \in X_\star(T^c)^{M,+}$ runs through the set $\{-ww_0(\nu+\rho)-\rho, w \in {}^MW\}$, with ${}^MW \subseteq W$ the set of minimal length representatives of $W_M\backslash W$.

In [CS17] and [CS19], Caraiani and Scholze proved a similar concentration result for the Betti cohomology of unitary Shimura varieties under a genericity condition for the action of the spherical Hecke algebra at a prime number p. Their result is much more powerful because it also applies to the cohomology with coefficients in a ℓ-torsion local system for a prime $\ell \neq p$. The three conditions of temperdness at infinity, genericity at p and small slope at p are related as the following example shows.

Example 1.12. Consider a compact Shimura curve S_K associated to a quaternion algebra over $\mathbb{Q}$ split at ∞ and p. Assume that $K = K^pK_p$ and $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$. Consider $1 \in \mathrm{H}^0(S_K, \mathscr{O}_{S_K})$, which comes from the trivial automorphic representation. The degree 0 cohomology is the "wrong" degree, in the sense that the interesting cohomology of $\mathscr{O}_{S_K} = \mathcal{V}_0$ sits in degree 1. We observe that the cohomology class 1 is:

(1) Non tempered at ∞, since $\mathrm{Vol}(\mathrm{PGL}_2(\mathbb{R})) = \infty$,
(2) Not small slope at p, because the U_p eigenvalue of the trivial representation is p, and $v(p) = 1$. The small slope condition in weight 0 is having U_p-eigenvalue of slope < 1.
(3) Not generic, because the semi-simple conjugacy class attached to the trivial representation via the local Langlands correspondence is $\mathrm{diag}(p^{\frac{1}{2}}, p^{-\frac{1}{2}})$.

We see that any of the conditions: tempered at infinity, generic at p, and small slope at p can be used to eliminate this class.

We now explain how these results are obtained. The main new object we introduce are local cohomology theories and we briefly explain how these theories are defined. Let $FL_{G,\mu} = P_\mu\backslash G$ be the flag variety attached to P_μ. There is a stratification into B-orbits, $FL_{G,\mu} = \coprod_{w \in {}^MW} P_\mu\backslash P_\mu wB$ where ${}^MW \subseteq W$ is the set of minimal length representatives of $W_M\backslash W$. Let $\mathcal{FL}_{G,\mu}$ be the associated adic space. Let $\mathcal{S}^{tor}_{K^p,\Sigma} = \lim_{K'_p \subseteq K_p} (S^{tor}_{K'_pK^p,\Sigma})^{ad}$ be the adic Shimura variety of infinite level at p (this is a perfectoid space at least if (G, X) is of Hodge type and Σ is well chosen). There is a Hodge-Tate period map $\pi^{tor}_{HT} : \mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{FL}_{G,\mu}$ defined in [Sch15] and [CS17].

Fix $w \in {}^MW$, $\kappa \in X^\star(T^c)^{M,+}$ and let $\chi : T(\mathbb{Z}_p) \to \overline{\mathbb{Q}}_p^\times$ be a finite order character. We define in section 5.4 local cohomology theories $\mathrm{R}\Gamma_w(K^p, \kappa, \chi)^{\pm, fs}$ and $\mathrm{R}\Gamma_w(K^p, \kappa, \chi, cusp)^{\pm, fs}$. These are cohomologies with support on certain open subsets of $\mathcal{S}^{tor}_{K_pK^p,\Sigma}$ for suitable K_p. These open subsets and support condition are defined with the help of the period map π^{tor}_{HT} and the Bruhat stratification. We also impose the finite slope condition, with respect to certain operators in Hecke algebra at p. The $\pm$ sign refers to two possible choices of Hecke operators at p, which are exchanged when we consider Serre duality.

Let us give some more detail on the support conditions. We can view ${}^MW$ as the set of $T(\mathbb{Q}_p)$-fixed point in $\mathcal{FL}_{G,\mu}$. For any $x \in FL_{G,\mu}(\mathbb{C}_p)$, $(\pi^{tor}_{HT})^{-1}(\{x\})$ is a perfectoid Igusa variety (see [CS17], sect. 4.3). Roughly speaking, the choice of x determines a p-divisible group with a trivialization of its Tate module (and extra structures) by [SW13], and $(\pi_{HT}^{tor})^{-1}(\{x\})$ is the space of all abelian varieties with that given p-divisible group. The subset ${}^MW \subseteq \mathcal{FL}_{G,\mu}$ corresponds to very particular p-divisible groups. For example, in the Siegel case, ${}^MW$ is contained in $\mathcal{FL}_{G,\mu}(\mathbb{Q}_p)$ and the corresponding p-divisible groups are ordinary. The support conditions can be read from the dynamics of the action of $T(\mathbb{Q}_p)$ near the fixed point w. In summary, the cohomologies $\mathrm{R}\Gamma_w(K^p,\kappa)^{\pm,fs}$ and $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{\pm,fs}$ are therefore cohomologies with support over some neighborhoods of the Igusa variety $(\pi_{HT}^{tor})^{-1}(\{w\})$.

Example 1.13. For $\mathrm{GL}_2/\mathbb{Q}$ and $w = Id$, $\mathrm{R}\Gamma_{Id}(K^p,\kappa,\chi)^{+,fs}$ is just (the finite slope part of) the space of weight κ and nebentypus χ overconvergent modular forms (viewed in degree 0). For w the only non-trivial element of the Weyl group, $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{+,fs}$ is concentrated in degree 1 and is the finite slope part of the cohomology with compact support of the "dagger" space ordinary locus. See [BP20].

Since Igusa varieties are affine in the minimal compactification, we can prove in theorem 5.18 that $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{\pm,fs}$ is concentrated in the range $[0,\ell_\pm(w)]$, with $\ell_+(w) = \ell(w)$ and $\ell_-(w) = d - \ell(w)$.

One of our main results is theorem 5.15 which is the existence of a spectral sequence which expresses the finite slope part $\mathrm{H}^i(K_p,\kappa,\chi)^{\pm,fs}$ of the classical cohomology in weight κ an nebentypus χ in term of the $\mathrm{H}^\star_w(K^p,\kappa,\chi)^{\pm,fs}$. This spectral sequence means therefore that the finite slope cohomology of the full Shimura variety can be understood with the help of the overconvergent cohomologies of the Igusa varieties corresponding to $w \in {}^MW$.

From the spectral sequence, we can actually extract a complex $\mathcal{C}ous(K^p,\kappa,\chi)^\pm$:

$$\mathrm{H}^0_{Id/w_0^M}(K^p,\kappa,\chi)^{\pm,fs} \to \cdots \to \oplus_{w\in {}^MW}\mathrm{H}^{\ell_\pm(w)}_w(K^p,\kappa,\chi)^{\pm,fs} \cdots \to \mathrm{H}^d_{w_0^M/Id}(K^p,\kappa,\chi)^{\pm,fs}$$

that is an analogue of the Grothendieck-Cousin complex which computes the cohomology of the flag varieties (here w_0^M is the longest element in ${}^MW$). We have also a cuspidal and interior variant of this complex: $\mathcal{C}ous(K^p,\kappa,\chi,cusp)^\pm$ and $\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^\pm$.

We conjecture (conjecture 5.20) that when the Shimura variety is compact, this complex computes $\mathrm{H}^i(K^p,\kappa,\chi)^{\pm,fs}$. For general Shimura varieties we can prove (corollary 5.27) that $\overline{\mathrm{H}}^i(K^p,\kappa,\chi)^{\pm,fs}$ is a subquotient of

$$\mathrm{H}^i(\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^\pm).$$

An important technical result is theorem 5.33 which bounds below the slopes for the action of Hecke operators at p on each of the cohomologies $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{\pm,fs}$, $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{\pm,fs}$. We deduce that $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{+,fs}$ has small slope vectors only if $w \in C(\kappa)^+ = \{w \in W, w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^-_{\mathbb{Q}}\}$ and that $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{-,fs}$ has small slope vectors only if $w \in C(\kappa)^- = \{w \in W, w^{-1}(\kappa+\rho) \in X^\star(T)^+_{\mathbb{Q}}\}$ (and similarly for cuspidal cohomology). This explains how we prove the cohomological vanishing by combining the cohomological vanishing for the $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{\pm,fs}$, the slope estimates on these cohomologies, and the spectral sequence converging to classical cohomology. This bound also implies a classicity theorem when $\kappa+\rho$ is regular and $C(\kappa)^\pm$ are reduced to a single element (many cases of this theorem for the degree 0 cohomology of PEL Shimura varieties were already proven, see for example [Col96], [Kas06], [Pil11], [BPS16], [TX16], [Bij17]):

Theorem 1.14 (Theorem 5.66). *Let $\kappa \in X^\star(T^c)^{M,+}$ be a weight such that $\kappa+\rho$ is G-regular. Then we have a quasi-isomorphism :*

$$\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{\pm,sss^M(\kappa)} = \mathrm{R}\Gamma(K^p,\kappa,\chi)^{\pm,sss^M(\kappa)}$$

where $\{w\} = C(\kappa)^{\pm}$, and similarly for cuspidal cohomology.

Remark 1.15. Here , $\pm, sss^M(\kappa)$ is a strongly small slope condition. We actually conjecture that a weaker small slope condition $\pm, ss^M(\kappa)$ would be sufficient.

The techniques of local cohomology can also be applied to de Rham cohomology. Let $\nu \in X^\star(T^c)^+$. Let $(\mathcal{W}^\vee_{\nu,dR},\nabla)$ the corresponding filtered vector bundle with logarithmic connection. We consider the finite slope part of de Rham cohomology groups in weight ν and nebentypus χ: $\mathrm{R}\Gamma_{dR}(K^p,\mathcal{W}^\vee_\nu,\chi)^{\pm,fs}$ and the de Rham cohomology with compact support $\mathrm{R}\Gamma_{dR,c}(K^p,\mathcal{W}^\vee_\nu,\chi)^{\pm,fs}$. We can construct local de Rham cohomology groups $\mathrm{R}\Gamma_{dR,w}(K^p,\mathcal{W}^\vee_\nu,\chi)^{\pm,fs}$ (and similarly for compactly supported cohomology), and we again have a spectral sequence from local to classical de Rham cohomology. We also have Falting's dual BGG resolution. We can combine everything to construct a double Cousin complex, as well as its compactly supported and interior versions. The interior double Cousin complex is:

$$(\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu,\chi)^{\pm})^{(i,j)} =$$
$$\oplus_{w\in {}^MW,\ell_\pm(w)=i}\oplus_{w'\in {}^MW,\ell(w')=j}\overline{\mathrm{H}}^i_w(K^p,-w'w_0(\nu+\rho)-\rho,\chi)^{\pm,fs}$$

whose horizontal line are the interior Cousin complexes of the automorphic vector bundles of weight $-w'w_0(\nu+\rho)-\rho$ and the vertical differentials are obtained from the differentials of the dual BGG complex of $\mathcal{W}^\vee_{\nu,dR}$. For compact Shimura varieties, we conjecture that the total complex associated with the double Cousin complex computes the de Rham cohomology. We can prove (proposition 5.89) that the interior de Rham cohomolgy groups $\overline{\mathrm{H}}^i_{dR}(K^p,\mathcal{W}^\vee_\nu,\chi)^{\pm}$ is a subquotient of $\mathrm{H}^i(\mathrm{Tot}(\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu,\chi)^{\pm}))$. Our main result for classical de Rham cohomology is the following decomposition (and vanishing) theorem for the strongly small slope part:

Theorem 1.16 (Theorem 5.87). *For all $\nu \in X^\star(T^c)^+$, we have that:*

$$\mathrm{H}^n_{dR}(K^p,\mathcal{W}^\vee_\nu,\chi)^{+,sss_b(\nu)} = \bigoplus_{p+q=n}\oplus_{w,\ell(w)=p}\mathrm{H}^q(K^p,-ww_0(\nu+\rho)-\rho,\chi)^{+,sss_b(\nu)}$$

and that

$$\mathrm{H}^n_{dR}(K^p,\mathcal{W}^\vee_\nu,\chi)^{-,sss_b(\nu)} = \bigoplus_{p+q=n}\oplus_{w,\ell_-(w)=p}\mathrm{H}^q(K^p,-ww_0(\nu+\rho)-\rho,\chi)^{-,sss_b(\nu)}$$

Moreover, these de Rham cohomology group vanish except if $n\in[d,2d]$. We have similarly:

$$\mathrm{H}^n_{dR,c}(K^p,\mathcal{W}^\vee_\nu,\chi)^{+,sss_b(\nu)} = \bigoplus_{p+q=n}\oplus_{w,\ell(w)=p}\mathrm{H}^q(K^p,-ww_0(\nu+\rho)-\rho,\chi,cusp)^{+,sss_b(\nu)}$$

and that

$$\mathrm{H}^n_{dR,c}(K^p,\mathcal{W}^\vee_\nu,\chi)^{-,sss_b(\nu)} = \bigoplus_{p+q=n}\oplus_{w,\ell_-(w)=p}\mathrm{H}^q(K^p,-ww_0(\nu+\rho)-\rho,\chi,cusp)^{-,sss_b(\nu)}.$$

Moreover, these de Rham cohomology group vanish except if $n\in[0,d]$.

Remark 1.17. This decomposition of de Rham cohomology is reminiscent of the complex decomposition using harmonic forms. We believe that this decomposition should be induced by a Frobenius action on the cohomology.

Our next results consider p-adic interpolation. In the classical work [CM98], the eigencurve was constructed by interpolating the Hecke action on the spaces of overconvergent modular forms $\mathrm{H}^0_{Id}(K^p,\kappa,\chi)^{+,fs}$ in the p-adic weight $\kappa\chi$. This approach of interpolating overconvergent modular forms was generalized to Siegel varieties in [AIP15] (see also [CHJ17], [Bra16], [Her19], [Bra20] for further generalizations). In another direction, p-adic interpolation of the cohomologies $\mathrm{H}^i(S_K(\mathbb{C}),\mathcal{W}_\nu^\vee)$ was considered (and in this setting, one considers general arithmetic quotient of locally symmetric space, not only Shimura varieties). See for example [AS08], [Urb11], [Han17], [Eme06]. We also mention the case of Shimura sets which has been studied intensively, and for which there is no distinction between Betti and coherent cohomology (see for example [Buz07], [Che04]). In this paper, we construct eigenvarieties by interpolating the local cohomologies $\mathrm{H}^n_w(K^p,\kappa,\chi)^{\pm,fs}$.

Remark 1.18. For the $\mathrm{H}^0_{Id}(K^p,\kappa,\chi)^{+,fs}$, we therefore recover certain of the constructions recalled above. The improvement is that we are not assuming that the group $G_{\mathbb{Q}_p}$ is unramified or that there is a dense ordinary locus.

By Faltings' BGG spectral sequence and its degeneration, there is a Hecke equivariant isomorphism

$$\mathrm{H}^i(S_K(\mathbb{C}),\mathcal{W}_\nu^\vee)=\oplus_{w\in {}^MW}\mathrm{H}^{i-d+\ell(w)}(S^{tor}_{K,\Sigma}(\mathbb{C}),\mathcal{V}_{-w_{0,M}w(\nu+\rho)-\rho})$$

as well as similar variants for cuspidal and interior cohomology. The point of view of this paper is to consider the p-adic interpolation of the right hand side of this expression. We also want to consider Serre duality. Thus, on the classical side, we are led to consider the finite slope cohomology groups (with nebentypus χ):

$$\mathrm{H}^\star(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,fs},\mathrm{H}^\star(K^p,w(\nu+\rho)-\rho,\chi^{-1})^{-,fs},$$

$$\mathrm{H}^\star(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi,cusp)^{+,fs},\mathrm{H}^\star(K^p,w(\nu+\rho)-\rho,\chi^{-1},cusp)^{-,fs}$$

together with the Serre duality pairing :

$$\mathrm{H}^i(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,fs}\times\mathrm{H}^{d-i}(K^p,w(\nu+\rho)-\rho,\chi^{-1},cusp)^{-,fs}\to\overline{\mathbb{Q}}_p$$

and

$$\mathrm{H}^i(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi,cusp)^{+,fs}\times\mathrm{H}^{d-i}(K^p,w(\nu+\rho)-\rho,\chi^{-1})^{-,fs}\to\overline{\mathbb{Q}}_p.$$

We will make the p-adic interpolation of all these groups in the p-adic weight $\nu\chi:T^c(\mathbb{Z}_p)\to\overline{\mathbb{Q}}_p^\times$ (which is roughly the infinitesimal character of automorphic representations contributing to these cohomology). In order to do this we first replace the classical cohomology group $\mathrm{H}^\star(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,fs}$ by the local cohomology $\mathrm{H}^\star_w(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,fs}$ with respect to w, and then further need to consider an analytic local cohomology $\mathrm{H}^\star_{w,an}(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,fs}$ (section 6.3). The difference between $\mathrm{H}^\star_w(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,fs}$ and $\mathrm{H}^\star_{w,an}(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,fs}$ is that in the first case we consider the overconvergent cohomology with support of a classical automorphic vector bundle, while in the second case, we consider the overconvergent cohomology with support of a Banach sheaf modeled on a principal series representation of the same weight as the automorphic vector bundle. We proceed similarly for $\mathrm{H}^\star(K^p,w(\nu+\rho)-\rho,\chi^{-1})^{-,fs}$,

$\mathrm{H}^\star(K^p, -w_{0,M}w(\nu+\rho)-\rho, \chi, cusp)^{+,fs}$, $\mathrm{H}^\star(K^p, w(\nu+\rho)-\rho, \chi^{-1}, cusp)^{-,fs}$ and decorate them with a subscript ${}_{w,an}$.

Remark 1.19. The passage from $\mathrm{H}^\star(K^p, -w_{0,M}w(\nu+\rho)-\rho, \chi)^{+,fs}$ to $\mathrm{H}^\star_{w,an}(K^p, -w_{0,M}w(\nu+\rho)-\rho, \chi)^{+,fs}$ (and not to $\mathrm{H}^\star_{w',an}(K^p, -w_{0,M}w(\nu+\rho)-\rho, \chi)^{+,fs}$ for $w' \neq w$) is motivated by theorem 1.14.

Let $\mathcal{W} = \mathrm{Spa}(\mathbb{Z}_p[[T^c(\mathbb{Z}_p)]], \mathbb{Z}_p[[T^c(\mathbb{Z}_p)]]) \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the space of characters of $T^c(\mathbb{Z}_p)$. Our interpolation result is the existence of an eigenvariety $\pi : \mathcal{E} \to \mathcal{W}$. Points of the eigenvariety are triples (λ_p, λ^S) where λ_p is a character of $T(\mathbb{Q}_p)$, S is the finite set of primes which contains p and all primes where K^p is not hyperspecial and λ^S is a system of eigenvalues for the prime to S spherical Hecke algebra. The projection to $\mathcal{W}$ is given by $\lambda_p \mapsto \nu = \lambda_p|_{T(\mathbb{Z}_p)}$. When λ_p has locally algebraic weight $\nu = \nu_{alg}\chi$ (for χ a finite order character and $\nu_{alg} \in X^\star(T)$, we let $\lambda_p^{sm} = \lambda_p \nu_{alg}^{-1}$.

Theorem 1.20 (Theorem 6.41, 6.42 and 6.45)**.** *The eigenvariety $\pi : \mathcal{E} \to \mathcal{W}$ is locally quasi-finite and partially proper. It carries graded coherent sheaves*

$$\bigoplus_{w \in {}^M W, k \in \mathbb{Z}} \Big(\mathrm{H}^k(\tilde{\mathcal{M}}_w^{\bullet,+,fs}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}_w^{\bullet,-,fs}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp}^{\bullet,+,fs}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp}^{\bullet,-,fs}) \Big)$$

and they satisfy the following properties:

1. *(Any classical, finite slope eigenclass gives a point of the eigenvariety) For any $\kappa_{alg} \in X^\star(T^c)^{M,+}$, finite order character $\chi : T(\mathbb{Z}_p) \to \overline{F}^\times$, and any system of Hecke eigenvalues (λ_p, λ^S) occurring in $\mathrm{H}^i(K^p, \kappa_{alg}, \chi)^{+,fs}$ (resp. $\mathrm{H}^i(K^p, -2\rho_{nc} - w_{0,M}\kappa_{alg}, \chi^{-1})^{-,fs}$, $\mathrm{H}^i(K^p, \kappa_{alg}, \chi, cusp)^{+,fs}$, or $\mathrm{H}^i(K^p, -2\rho_{nc} - w_{0,M}\kappa_{alg}, \chi^{-1}, cusp)^{-,fs}$) there is a $w = w_M w^M \in W$, so that if $\nu = \nu_{alg}\chi$ with $\nu_{alg} = -w^{-1}w_{0,M}(\kappa_{alg}+\rho)-\rho$, then $(\nu_{alg}\lambda_p, \lambda^S)$ is a point of the eigenvariety $\mathcal{E}$ which lies in the support of $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_{w^M}^{\bullet,+,fs})$ (resp. $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_{w^M}^{\bullet,-,fs})$, $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_{w^M,cusp}^{\bullet,+,fs})$, or $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_{w^M,cusp}^{\bullet,-,fs})$).*
2. *(Small slope points of the eigenvariety in regular, locally algebraic weights are classical) Conversely if $\nu = \nu_{alg}\chi$ is a locally algebraic weight with $\nu_{alg} \in X^\star(T^c)^+$, and $(\nu, \lambda_p, \lambda^S)$ is a point of $\mathcal{E}$ in the support of $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_w^{\bullet,+,fs})$ (resp. $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_w^{\bullet,-,fs})$, $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_{w,cusp}^{\bullet,+,fs})$, or $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}_{w,cusp}^{\bullet,-,fs})$) for some $w \in {}^M W$, and if λ_p satisfies $+, sss_w(\nu)$ then $(\lambda_p^{sm}, \lambda^S)$ occurs in $\mathrm{H}^i(K^p, \kappa_{alg}, \chi)^{+,fs}$ (resp. $\mathrm{H}^i(K^p, -2\rho_{nc} - w_{0,M}\kappa_{alg}, \chi^{-1})^{-,fs}$, $\mathrm{H}^i(K^p, \kappa_{alg}, \chi, cusp)^{+,fs}$, or $\mathrm{H}^i(K^p, -2\rho_{nc} - w_{0,M}\kappa_{alg}, \chi^{-1}, cusp)^{-,fs}$) for $\kappa_{alg} = -w_{0,M}w(\nu_{alg}+\rho)-\rho$ and some i.*
3. *(Serre duality interpolates over the eigenvariety) We have pairings:*
$$\mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp}^{\bullet,\pm,fs}) \otimes \mathrm{H}^{d-k}(\tilde{\mathcal{M}}_w^{\bullet,\mp,fs}) \to \pi^{-1}\mathscr{O}_{\mathcal{W}}.$$
and these pairings are compatible with Serre duality under the classicality theorem.
4. *(Interior cohomology class deform over the weight space) Any interior cohomology class $c \in \overline{\mathrm{H}}^\star(K^p, \chi, \kappa)^{-,fs}$ belongs to a component of the eigenvariety of dimension equal to the dimension of the weight space.*

Remark 1.21. The condition $+, sss_w(\nu)$ is a strongly small slope condition.

Remark 1.22. It is plausible that the eigenvariety $\mathcal{E}$ coincides with an eigenvariety constructed via Betti cohomology interpolation, as in [Han17]. One can also believe that there is a p-adic Eichler-Shimura theory relating both constructions. See [AIS15], as well as some forthcoming work of Juan Esteban Rodriguez.

Remark 1.23. We can define certain $\pi^{-1}\mathscr{O}_{\mathcal{W}}$-torsion free sheaves $\overline{\mathcal{C}ous}_w^{\pm}$ over the eigenvariety. The sheaves $\{\overline{\mathcal{C}ous}_w^{\pm}\}_w$ interpolate the various modules of the interior Cousin complex (which can be used to compute the interior cohomology). For all $w \in {}^MW$, we let $\mathcal{E}^!_w$ be the support of $\overline{\mathcal{C}ous}_w^{\pm}$, which is a union of irreducible components of the eigenvariety of dimension equal to dim $\mathcal{W}$. Any interior cohomology class lifts to a point on $\cup_{w\in {}^MW}\mathcal{E}^!_w$.

Remark 1.24. Any classical interior cohomology eigenclass $c \in \overline{\mathrm{H}}^\star(K^p,\kappa,\chi)^{-,fs}$ gives a point on the eigenvariety, and we can attach to it the finite subset ${}^MW(c)$ of $w \in {}^MW$ such that $c \in \mathcal{E}^!_w$. It seems interesting to describe this set. We can prove (proposition 6.46) that under certain slope assumptions, we have ${}^MW(c) \subseteq w_{0,M}C(\kappa)^+w_0$.

Remark 1.25. For the group $\mathrm{GL}_2/\mathbb{Q}$, the theory was explained in [BP20]. In this case $\mathcal{E}$ is already equidimensional of dimension 1, and is the Coleman-Mazur eigencurve. The sheaves $\overline{\mathcal{C}ous}_w^{+}$ and $\overline{\mathcal{C}ous}_w^{-}$ are in perfect duality, and similarly $\overline{\mathcal{C}ous}_1^{+}$ and $\overline{\mathcal{C}ous}_1^{-}$ are also in perfect duality. Furthermore, the Atkin-Lehner involution induces an isomorphism between $\overline{\mathcal{C}ous}_{Id/w}^{\pm}$ and $\overline{\mathcal{C}ous}_{w/Id}^{\mp}$. It follows that $\mathcal{E}^!_1 = \mathcal{E}^!_w$ is the cuspidal part of the eigenvariety.

Remark 1.26. For GSp_4/L with L a totally real field, variants or special cases of this theory are considered in [Pil20], [BCGP18], [LPSZ19].

Remark 1.27. The point (4) of the theorem is an advantage of the method we use to construct the eigenvarieties. Such a result was only available in a limited number of cases (Shimura sets, automorphic forms contributing to cuspidal coherent H^0...).

Finally, using point (4) we can give a new construction of the Galois representations of certain automorphic forms realizing in the coherent cohomology of Shimura varieties but not in the Betti cohomology. This construction is via analytic interpolation, and yields results on local-global compatibility at p. In [FP19], section 9, we defined a certain class of cuspidal automorphic forms for the group GL_n/L where L is either a totally real or CM number field. These are called weakly regular, odd, essentially conjugate self dual algebraic cuspidal automorphic representations.

Theorem 1.28. *Let π be a weakly regular, algebraic, odd, (essentially) conjugate self dual, cuspidal automorphic representation of GL_n/L. In particular, $\pi^c = \pi^\vee \otimes \chi$. Let $\lambda = (\lambda_{i,\tau}, 1 \leq i \leq n, \tau \in \mathrm{Hom}(L,\overline{\mathbb{Q}}))$ and $\lambda_{1,\tau} \geq \cdots \geq \lambda_{n,\tau}$ be the infinitesimal character of π. There is a continuous Galois representation $\rho_{\pi,\iota} : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that:*

1. *$\rho^c_{\pi,\iota} \simeq \rho^\vee \otimes \epsilon_p^{1-n} \otimes \chi_\iota$ where χ_ι is the p-adic realization of χ,*
2. *$\rho_{\pi,\iota}$ is unramified at all finite places $v \nmid p$ for which π_v is unramified and one has:*
$$WD(\rho_{\pi,\iota}|_{G_{L_v}})^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$
3. *$\rho_{\pi,\iota}$ has generalized Hodge–Tate weights $(-\lambda_{n,\tau} + \frac{n-1}{2}, \cdots, -\lambda_{1,\tau} + \frac{n-1}{2})$.*

(4) *Let $v \mid p$ be a place of L and assume that π_v is an irreducible unramified principal series representation with distinct Satake parameters. Then $\rho_{\pi,\iota}|_{G_{L_v}}$ is crystalline and*

$$WD(\rho_{\pi,\iota}|_{G_{L_v}})^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$

Remark 1.29. If π is regular rather than weakly regular, then a stronger form of the theorem holds according to results of Bellaiche, Caraiani, Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin, Taylor... (see [CH13], [BLGGT14]) including purity and local-global compatibility at all places. The above theorem is deduced from these results by p-adic analytic interpolation.

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2. Cohomological preliminaries

2.1. **Cohomology with support in a closed subspace.** We recall the notion and the basic properties of the cohomology of an abelian sheaf on a topological space, with support in a closed subspace. A reference for this material is [Gro05], chapter I. Let X be a topological space. We let Ab_X be the category of abelian sheaves over X. If X is a point, Ab_X is simply Ab the category of abelian groups. We let $\mathcal{D}(Ab_X)$ be the derived category of Ab_X. Let $i : Z \hookrightarrow X$ be a closed subspace. For an object $\mathscr{F}$ of Ab_X, we let $\Gamma_Z(X, \mathscr{F})$ be the subgroup of $\mathrm{H}^0(X, \mathscr{F})$ of sections whose support is included in Z. We let $\mathrm{R}\Gamma_Z(X, -) : \mathcal{D}(Ab_X) \to \mathcal{D}(Ab)$ be the right derived functor of $\Gamma_Z(X, -)$ (see [Sta13], Tag 079V, in particular for the unbounded version). Let $U = X \setminus Z$ and let $\mathscr{F}$ be an object of $\mathcal{D}(Ab_X)$. We have an exact triangle in $\mathcal{D}(Ab)$ ([Gro05], I, corollaire 2.9):

$$\mathrm{R}\Gamma_Z(X, \mathscr{F}) \to \mathrm{R}\Gamma(X, \mathscr{F}) \to \mathrm{R}\Gamma(U, \mathscr{F}) \stackrel{+1}{\to}$$

We have the classical pushforward functor $i_\star : Ab_Z \to Ab_X$ and it admits a right adjoint $i^! : Ab_X \to Ab_Z$ which can be described as follows: Let $W \subseteq Z$ be an open subset. Let $W' \subseteq X$ be an open subset of X such that $W = W' \cap Z$. For any object $\mathscr{F}$ of Ab_X, we have $i^!\mathscr{F}(W) = \Gamma_W(W', \mathscr{F}|_{W'})$. It follows that $\Gamma_Z(X, \mathscr{F}) = \mathrm{H}^0(X, i_\star i^! \mathscr{F})$. The functor $i^!$ has a right derived functor $\mathrm{R}i^! : \mathcal{D}(Ab_X) \to \mathcal{D}(Ab_Z)$. Moreover $\mathrm{R}\Gamma_Z(X, \mathscr{F}) = \mathrm{R}\Gamma(X, i_\star \mathrm{R}i^! \mathscr{F})$.

Some properties of the cohomology with support are:

(1) (change of support)[[Gro05], I, Proposition 1.8] If $Z \subseteq Z'$, there is a map $\mathrm{R}\Gamma_Z(X, \mathscr{F}) \to \mathrm{R}\Gamma_{Z'}(X, \mathscr{F})$.
(2) (pull-back) If we have a cartesian diagram:

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z' & \longrightarrow & X' \end{array}$$

and a sheaf $\mathscr{F}$ on X', there is a map $\mathrm{R}\Gamma_{Z'}(X', \mathscr{F}) \to \mathrm{R}\Gamma_Z(X, f^\star \mathscr{F})$.

(3) (Change of ambient space)[[Gro05], I, Proposition 2.2] If we have $Z \subset U \subset X$ for some open U of X, then the pull back map $\mathrm{R}\Gamma_Z(X, \mathscr{F}) \to \mathrm{R}\Gamma_Z(U, \mathscr{F})$ is a quasi-isomorphism.

The above properties imply easily the following lemma:

Lemma 2.1. *Let $Z_1, Z_2 \subset X$ be two disjoint closed subsets. Then the natural map given by pushforward:*

$$\mathrm{R}\Gamma_{Z_1}(X, \mathscr{F}) \oplus \mathrm{R}\Gamma_{Z_2}(X, \mathscr{F}) \to \mathrm{R}\Gamma_{Z_1 \cup Z_2}(X, \mathscr{F})$$

is a quasi-isomorphism.

Proof. For $i \in \{1, 2\}$, we let $i_i : Z_i \hookrightarrow X$. We finally let $i : Z_1 \cup Z_2 \hookrightarrow X$. The lemma will follow from the claim that for any $\mathscr{F} \in Ob(C_X)$, the natural map $(i_1)_\star i_1^! \mathscr{F} \oplus (i_2)_\star i_2^! \mathscr{F} \to i_\star i^! \mathscr{F}$ is an isomorphism of sheaves. This is a local computation. Since $X = Z_1^c \cup Z_2^c$ and the claim holds true over any open subset of Z_1^c or Z_2^c, we are done. □

We now discuss the construction of the trace map in the context of schemes or adic spaces and finite locally free morphisms ([Hub96], Sect. 1.4.4).

Lemma 2.2. *Consider a commutative diagram of topological spaces:*

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z' & \longrightarrow & X' \end{array}$$

with X and X' ringed spaces, f a finite locally free morphism of schemes or adic spaces, $Z' \to X'$ and $Z \to X$ closed subspaces. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_{X'}$-modules. Then there is a map $\mathrm{R}\Gamma_Z(X, f^\star \mathscr{F}) \to \mathrm{R}\Gamma_{Z'}(X', \mathscr{F})$.

Proof. We first recall that the category of sheaves of $\mathscr{O}_T$-modules on a ringed space $(T, \mathscr{O}_T)$ has enough injectives ([Sta13], Tag 01DH). It follows that it is enough to construct a functorial map $\Gamma_Z(X, f^\star \mathscr{F}) \to \Gamma_{Z'}(X', \mathscr{F})$ for sheaves $\mathscr{F}$ of $\mathscr{O}_X$-modules. If we let $Z'' = f^{-1}(Z')$, then we have a map $\Gamma_Z(X, f^\star \mathscr{F}) \to \Gamma_{f^{-1}(Z')}(X, f^\star \mathscr{F})$. Therefore, it suffices to consider the case where $Z = f^{-1}(Z')$. We have a trace map $\mathrm{Tr} : f_\star \mathscr{O}_X \to \mathscr{O}_{X'}$. Moreover, the natural morphism $f_\star \mathscr{O}_X \otimes_{\mathscr{O}_{X'}} \mathscr{F} \to f_\star f^\star \mathscr{F}$ is an isomorphism. We therefore have a trace map $\mathrm{Tr} : f_\star f^\star \mathscr{F} \to \mathscr{F}$. Let us complete the above diagram into:

$$\begin{array}{ccccc} Z & \longrightarrow & X & \longleftarrow & U \\ \downarrow & & \downarrow f & & \downarrow g \\ Z' & \longrightarrow & X' & \overset{j'}{\longleftarrow} & U' \end{array}$$

where $U' = X' \setminus Z'$ and $U = X \setminus Z$. We have a commutative diagram:

$$\begin{array}{ccc} f_\star f^\star \mathscr{F} & \longrightarrow & j'_\star g_\star g^\star (j')^\star \mathscr{F} \\ \Big\downarrow{\scriptstyle \mathrm{Tr}} & & \Big\downarrow \\ \mathscr{F} & \longrightarrow & j'_\star (j')^\star \mathscr{F} \end{array}$$

We deduce that the trace map $\Gamma(X, f^\star \mathscr{F}) \to \Gamma(X', \mathscr{F})$ induces a trace map $\Gamma_Z(X, f^\star \mathscr{F}) \to \Gamma_{Z'}(X', \mathscr{F})$. □

2.2. **Cup products.** Let $(X, \mathscr{O}_X)$ be a ringed space. Let $K = \Gamma(X, \mathscr{O}_X)$.

Proposition 2.3. *Let $Z_1, Z_2 \subseteq X$ be two closed subsets. Let $\mathscr{F}$ and $\mathscr{G}$ be two flat sheaves of $\mathscr{O}_X$-modules. There is a map:*

$$\mathrm{R}\Gamma_{Z_1}(X, \mathscr{F}) \otimes_K^L \mathrm{R}\Gamma_{Z_2}(X, \mathscr{G}) \to \mathrm{R}\Gamma_{Z_1 \cap Z_2}(X, \mathscr{F} \otimes_{\mathscr{O}_X}^L \mathscr{G}).$$

Proof. Let $Z_3 = Z_1 \cap Z_2$. For $1 \leq j \leq 3$, we let $i_j : Z_i \hookrightarrow X$ be the inclusion. For $\mathscr{F}$ and $\mathscr{G}$ sheaves of $\mathscr{O}_X$-modules we have a map $(i_1)_\star i_1^! \mathscr{F} \otimes_{\mathscr{O}_X} (i_2)_\star i_2^! \mathscr{G} \to (i_3)_\star i_3^! (\mathscr{F} \otimes \mathscr{G})$. We claim that there is a map in the derived category:

$$(i_1)_\star \mathrm{R} i_1^! \mathscr{F} \otimes_{\mathscr{O}_X}^L (i_2)_\star \mathrm{R} i_2^! \mathscr{G} \to (i_3)_\star \mathrm{R} i_3^! (\mathscr{F} \otimes \mathscr{G}).$$

Indeed, taking the Godement resolution ([Sta13], Tag 0FKR) gives quasi-isomorphisms $\mathscr{F} \to \mathscr{F}^\bullet$ and $\mathscr{G} \to \mathscr{G}^\bullet$ where $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ are bounded below complexes of flasque sheaves of $\mathscr{O}_X$-modules and for all $x \in X$, the maps $\mathscr{F}_x \to \mathscr{F}_x^\bullet$ and $\mathscr{G}_x \to \mathscr{G}_x^\bullet$ are homotopy equivalences in the category of $\mathscr{O}_{X,x}$-modules. In particular, $\mathscr{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat (by [Sta13], Lemma 06YB and the property that $\mathscr{F}_x$ and $\mathscr{G}_x$ are flat $\mathscr{O}_X$-modules). Since $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ are K-flat, the map $\mathscr{F} \otimes \mathscr{G} \to \mathrm{Tot}\big(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet\big)$ is a quasi-isomorphism. We see that $(i_1)_\star \mathrm{R} i_1^! \mathscr{F}$ is computed by $(i_1)_\star i_1^! \mathscr{F}^\bullet$ and $(i_2)_\star \mathrm{R} i_2^! \mathscr{G}$ is computed by $(i_2)_\star i_2^! \mathscr{G}^\bullet$. Taking K-flat resolutions $A^\bullet \to (i_1)_\star i_1^! \mathscr{F}^\bullet$ and $B^\bullet \to (i_2)_\star i_2^! \mathscr{G}^\bullet$, we see that $\mathrm{Tot}(A^\bullet \otimes_{\mathscr{O}_X} B^\bullet)$ computes

$$(i_1)_\star \mathrm{R} i_1^! \mathscr{F} \otimes_{\mathscr{O}_X}^L (i_2)_\star \mathrm{R} i_2^! \mathscr{G}$$

and there is a map

$$\mathrm{Tot}(A^\bullet \otimes_{\mathscr{O}_X} B^\bullet) \to \mathrm{Tot}\big((i_1)_\star i_1^! \mathscr{F}^\bullet \otimes_{\mathscr{O}_X} (i_2)_\star i_2^! \mathscr{F}^\bullet\big)$$

$$\to (i_3)_\star i_3^! (\mathrm{Tot}\big(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet\big)).$$

Taking a K-injective resolution $\mathrm{Tot}(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet) \to C^\bullet$ we finally find that $(i_3)_\star i_3^! (C^\bullet)$ computes $(i_3)_\star \mathrm{R} i_3^! (\mathscr{F} \otimes \mathscr{G})$ and we have a morphism

$$(i_3)_\star i_3^! (\mathrm{Tot}\big(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet\big)) \to (i_3)_\star i_3^! (C^\bullet).$$

There is also a usual cup-product map by [Sta13], Tag 0FPJ:

$$\mathrm{R}\Gamma(X, (i_1)_\star \mathrm{R} i_1^! \mathscr{F}) \otimes_K^L \mathrm{R}\Gamma(X, (i_2)_\star \mathrm{R} i_2^! \mathscr{G}) \to \mathrm{R}\Gamma(X, (i_1)_\star \mathrm{R} i_1^! \mathscr{F} \otimes_{\mathscr{O}_X}^L (i_2)_\star \mathrm{R} i_2^! \mathscr{G}).$$

Combining the two maps gives the map of the proposition. □

2.3. **The spectral sequence of a filtered topological space.** Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and let $W \subseteq Z$ be two closed subspaces of X. We can define $\mathrm{R}\Gamma_{Z/W}(X, \mathscr{F}) = \mathrm{R}\Gamma_{Z\setminus W}(X \setminus W, \mathscr{F})$. If $Z \subseteq Z'$ and $W \subseteq W'$ we have a map $\mathrm{R}\Gamma_{Z/W}(X, \mathscr{F}) \to \mathrm{R}\Gamma_{Z'/W'}(X, \mathscr{F})$.

If we have $Z_3 \subset Z_2 \subset Z_1$, then there is an exact triangle ([Kem78], lemma 7.6): $\mathrm{R}\Gamma_{Z_2/Z_3}(X, \mathscr{F}) \to \mathrm{R}\Gamma_{Z_1/Z_3}(X, \mathscr{F}) \to \mathrm{R}\Gamma_{Z_1/Z_2}(X, \mathscr{F}) \stackrel{+1}{\to}$

Assume that there is a filtration by closed subsets $X = Z_0 \supseteq Z_1 \supseteq \cdots Z_r = \emptyset$. Then there is a spectral sequence of filtered topological space ([Har66], p. 227):

$$E_1^{pq} = \mathrm{H}^{p+q}_{Z_p/Z_{p+1}}(X, \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$$

which we can visualize as follows:

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\mathrm{H}^1_{Z_0/Z_1}(X, \mathscr{F}) & \mathrm{H}^2_{Z_1/Z_2}(X, \mathscr{F}) & \cdots \\
\mathrm{H}^0_{Z_0/Z_1}(X, \mathscr{F}) & \mathrm{H}^1_{Z_1/Z_2}(X, \mathscr{F}) & \cdots \\
 & \mathrm{H}^0_{Z_1/Z_2}(X, \mathscr{F}) & \cdots
\end{array}$$

The differential $d_1^{p,q} : \mathrm{H}^{p+q}_{Z_p/Z_{p+1}}(X, \mathscr{F}) \to \mathrm{H}^{p+q+1}_{Z_{p+1}/Z_{p+2}}(X, \mathscr{F})$ is the boundary map in the long exact sequence associated with the triangle:

$$\mathrm{R}\Gamma_{Z_{p+1}/Z_{p+2}}(X, \mathcal{F}) \to \mathrm{R}\Gamma_{Z_p/Z_{p+2}}(X, \mathcal{F}) \to \mathrm{R}\Gamma_{Z_p/Z_{p+1}}(X, \mathcal{F}) \stackrel{+1}{\to} .$$

2.4. **The category of projective Banach modules.** In this work we will consider cohomologies that will be naturally represented by complexes of Banach modules (or projective limit of such complexes). We therefore recall the basics of this theory. Our discussion follows [Urb11], section 2.

2.4.1. *The derived category.* Let (A, A^+) be a complete Tate algebra over a non-archimedean field $(F, \mathcal{O}_F)$. We let $\varpi \in \mathcal{O}_F$ be a pseudo-uniformizer. A Banach A-module M is a topological A-module whose topology can be described as follows: Let A_0 be an open and bounded subring of A. Then M contains an open and bounded sub A_0-module M_0 which is ϖ-adically complete and separated. We let $\mathbf{Ban}(A)$ be the category of Banach A-modules. This is an exact category and one can consider its derived category $\mathcal{D}(\mathbf{Ban}(A))$ ([Urb11], sect. 2.1.3). Let I be a set. Denote $A(I)$ the submodule of A^I of sequences of sequences of elements of A indexed by I converging to 0 according to the filter in I of the complement of the finite subsets of I. This module can also be described as follows. Let A_0 be an open and bounded subring of A. Let $A_0(I)$ be the ϖ-adic completion of the free A_0-module with basis I. Then $A(I) = A_0(I)[\frac{1}{\varpi}]$. We see that $A(I)$ is a Banach A-module. A Banach A-module M is orthonormalizable if there exists a set I and an isomorphism $M \simeq A(I)$. A Banach A-module is called projective if it is a direct factor of an orthonormalizable Banach A-module. We let $\mathcal{K}^{proj}(A)$ be the category whose objects are bounded complexes of projective Banach A modules, and morphisms are homotopy classes of morphisms of such complexes. There is a natural functor $\mathcal{K}^{proj}(A) \to \mathcal{D}(\mathbf{Ban}(A))$ and it is fully faithful ([Urb11], lem. 2.1.8). Finally, we let $\mathcal{K}^{perf}(A)$ be the category whose objects are bounded complexes of finite projective A-modules, and morphisms are homotopy classes of morphisms of such complexes. The objects of this category are called perfect complexes. This is a full subcategory of $\mathcal{K}^{proj}(A)$.

We let $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$ be the category whose objects are projective system of complexes $\{K_i \in Ob(\mathcal{K}^{proj}(A))\}_{i\in\mathbb{N}}$ and the K_i's have non-zero cohomology in a uniformly bounded range of degrees. We denote an object of this category by "$\lim_{i\in\mathbb{N}}$"K_i

We end this section with a simple lemma:

Lemma 2.4. *Assume that A is noetherian. Let M be a projective Banach A-module. Then M is A-flat.*

Proof. We reduce to the case where $M = A(I)$ for a set I. We claim that for any finitely generated A-module N, $N \otimes_A A(I) = N(I)$ is the module of sequences of elements of N indexed by I and converging to 0. This implies that $A(I)$ is flat. Let $A^n \stackrel{a}{\to} A^m \stackrel{b}{\to} N \to 0$ be a presentation of N. Let $A_0 \subseteq A$ be an open and bounded sub-module. Then $b(A_0^m) = N_0$ is an open and bounded sub-module of N. For any $k \in \mathbb{Z}$, we find that if $y \in \varpi^k N_0$, then there is $x \in \varpi^k A_0^n$, $b(x) = y$. It follows that the map $A^n(I) \to N(I)$ is surjective. There is $l \in \mathbb{Z}$ such that $(\varpi^l A_0^m) \cap \mathrm{Ker}(b) \subseteq a(A_0^n)$. It follows that if $x \in \varpi^k A_0^m$ is such that $b(x) = 0$, then there is $y \in \varpi^{k-l} A_0^n$ such that $a(y) = x$. It follows that the map $A^n(I) \to \mathrm{Ker}(A^m(I) \to N(I))$ is surjective. We deduce that $N(I) = \mathrm{Coker}(A^n(I) \to A^m(I)) = N \otimes_A A(I)$. □

2.4.2. *Compact operators.* Recall that a continuous morphism $T : M \to N$ between Banach A-modules is called compact if it is a limit of finite rank operators (for the supremum norm of operators). Let $M^\bullet, N^\bullet \in Ob(\mathcal{K}^{proj}(A))$ and let

$$T \in \mathrm{Hom}_{\mathcal{D}(\mathbf{Ban}(A))}(M^\bullet, N^\bullet).$$

We say that T is compact if it has a representative $\tilde{T} \in \mathrm{Hom}_A(M^\bullet, N^\bullet)$ such that $\tilde{T}$ is compact in each degree.

Definition 2.5. *Let $M^\bullet \in Ob(\mathcal{K}^{proj}(A))$ and let $T \in \mathrm{End}_{\mathcal{D}(\mathbf{Ban}(A))}(M^\bullet)$. We say that T is potent compact if for some $n \geq 0$, T^n is compact.*

We need to extend these definitions to the case of objects in $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$. Let "$\lim_i$"$M_i^\bullet$, "$\lim_i$"$N_i^\bullet \in Ob(\mathrm{Pro}(\mathcal{K}^{proj}(A)))$ and let

$$T \in \mathrm{Hom}_{\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))}(\text{“}\lim_i\text{”} M_i^\bullet, \text{“}\lim_i\text{”} N_i^\bullet).$$

We say that T is compact if there exists $M^\bullet, N^\bullet \in Ob(\mathcal{K}^{proj}(A))$, a compact operator $T' \in \mathrm{Hom}_{\mathcal{D}(\mathbf{Ban}(A))}(M^\bullet, N^\bullet)$, and a commutative diagram:

$$\begin{array}{ccc} M^\bullet & \xrightarrow{T'} & N^\bullet \\ \uparrow & & \downarrow \\ \text{“}\lim_i\text{”} M_i^\bullet & \xrightarrow{T} & \text{“}\lim_i\text{”} N_i^\bullet \end{array}$$

We similarly say that T is potent compact if there exists a diagram as before with T' potent compact.

Lemma 2.6. *Let* "$\lim_i$"$M_i^\bullet \in Ob(\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A)))$ *and let* T *be a compact endomorphism of* "$\lim_i$"$M_i^\bullet$. *Then* T *induces canonically a compact endomorphism* T_i *of* $M_i^\bullet$ *for* i *large enough and there are factorization diagrams:*

$$\begin{array}{ccc} M_{i+1}^\bullet & \xrightarrow{T_{i+1}} & M_{i+1}^\bullet \\ \downarrow & \nearrow & \downarrow \\ M_i^\bullet & \xrightarrow{T_i} & M_i^\bullet \end{array}$$

Proof. By definition, the map "$\lim_i$"$M_i^\bullet \to M^\bullet$ factors into "$\lim_i$"$M_i^\bullet \to M_i \to M^\bullet$ for some i large enough. The map $N^\bullet \to$ "$\lim_i$"$M_i^\bullet$ is given by a collection of compatible maps $N^\bullet \to M_i^\bullet$. The lemma follows. □

2.5. **The cohomology of Banach sheaves.** In this section we explain how we can obtain complexes of Banach modules in the cohomology of rigid analytic varieties. We use the theory of adic spaces described in [Hub96] and [Hub94] for example.

2.5.1. *Sheaves of Banach modules over adic spaces.* In this section we recall some material from [AIP15], appendix A. Let F be a non archimedean field with ring of integers $\mathcal{O}_F$. Let $\varpi \in F$ be a topologically nilpotent unit.

We recall that there is a good theory of coherent sheaves on finite type adic spaces over $\mathrm{Spa}(F, \mathcal{O}_F)$. If $\mathcal{X} = \mathrm{Spa}(A, A^+)$ is affinoid and $\mathscr{F}$ is a coherent sheaf on X, then $\mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) = 0$ for $i \neq 0$, $M = \mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is an A-module of finite type and the canonical map $M \otimes_A \mathscr{O}_{\mathcal{X}} \to \mathscr{F}$ is an isomorphism ([Hub94], thm. 2.5). Moreover, M is canonically a Banach A-module ([Hub94], lem. 2.4). It follows that a coherent sheaf $\mathscr{F}$ over a finite type adic space $\mathcal{X}$ is a sheaf of topological $\mathscr{O}_{\mathcal{X}}$-modules. In this paper we will have to manipulate topological sheaves which are not coherent.

Definition 2.7. *Let* $\mathcal{X}$ *be a finite type adic space over* $\mathrm{Spa}(F, \mathcal{O}_F)$. *A sheaf* $\mathscr{F}$ *of topological* $\mathscr{O}_{\mathcal{X}}$*-modules is called a Banach sheaf if:*

1. *For any quasi-compact open* $\mathcal{U} \hookrightarrow \mathcal{X}$, $\mathscr{F}(\mathcal{U})$ *is a Banach* $\mathscr{O}_{\mathcal{X}}(\mathcal{U})$*-module,*
2. *There is an affinoid covering* $\mathcal{X} = \cup_i \mathcal{U}_i$, *such that for any affinoid* $\mathcal{V} \subset \mathcal{U}_i$, *the continuous restriction map* $\mathscr{O}_{\mathcal{X}}(\mathcal{V}) \otimes_{\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)} \mathscr{F}(\mathcal{U}_i) \to \mathscr{F}(\mathcal{V})$ *induces a topological isomorphism:* $\mathscr{O}_{\mathcal{X}}(\mathcal{V}) \hat{\otimes}_{\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)} \mathscr{F}(\mathcal{U}_i) \to \mathscr{F}(\mathcal{V})$,

A Banach sheaf $\mathscr{F}$ *is called projective if there is a covering as in* (2) *such that* $\mathscr{F}(\mathcal{U}_i)$ *is a projective Banach* $\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)$*-module.*

Any coherent sheaf on X is therefore a Banach sheaf and a coherent sheaf is a projective Banach sheaf if and only if it is projective. Banach sheaves over $\mathcal{X}$ form a full subcategory of the category of topological $\mathscr{O}_{\mathcal{X}}$-modules. We introduce compact morphisms in this context.

Definition 2.8. *Let* $\mathcal{X}$ *be an adic space of finite type over* $\mathrm{Spa}(F, \mathcal{O}_F)$. *Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be two projective Banach sheaves. Let* $\phi : \mathscr{F} \to \mathscr{G}$ *be a continuous morphism of* $\mathscr{O}_{\mathcal{X}}$*-modules. We say that the map* ϕ *is compact if there is a covering* $\mathcal{X} = \cup_i \mathcal{U}_i$ *satisfying the point* (2) *of definition 2.7 for both* $\mathscr{F}$ *and* $\mathscr{G}$, *such that the map* $\phi : \mathscr{F}(\mathcal{U}_i) \to \mathscr{G}(\mathcal{U}_i)$ *is a compact map of* $\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)$*-modules.*

Note that if $\mathscr{G}$ is coherent, any morphism to $\mathscr{G}$ is compact.

2.5.2. *Cohomological properties of Banach sheaves.* We warn the reader that Banach sheaves which are not coherent sheaves are pathological in general. In particular it is not true that for $\mathcal{X}$ affinoid a Banach sheaf has trivial higher cohomology groups or that a Banach sheaf is the sheaf associated with its global sections. Here is nevertheless a simple example. Let $\mathcal{X} = \mathrm{Spa}(A, A^+)$ be an affinoid and let M be a projective Banach A-module. Let $\mathscr{F} = M\hat{\otimes}_A\mathscr{O}_{\mathcal{X}}$ be the pre-sheaf whose value on an affinoid open $\mathcal{U} = \mathrm{Spa}(B, B^+)$ of $\mathcal{X}$ is $M\hat{\otimes}_A B$.

Lemma 2.9. *The pre-sheaf $\mathscr{F}$ is a sheaf and $\mathrm{H}^i(\mathcal{X}, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. We reduce to the case that M is orthonormalizable, and everything follows from the known properties of $\mathscr{O}_{\mathcal{X}}$. □

We now introduce a certain class of Banach sheaves that have better cohomological properties. These are Banach sheaves admitting formal models which can be controlled in a certain sense. We thus begin by discussing formal Banach sheaves over formal schemes.

Definition 2.10. *Let $\mathfrak{X} \to \mathrm{Spf}\ \mathcal{O}_F$ be a finite type formal scheme over $\mathrm{Spf}(\mathcal{O}_F)$. A sheaf $\mathfrak{F}$ of $\mathscr{O}_{\mathfrak{X}}$-modules is called a formal Banach sheaf if $\mathfrak{F}$ is flat as an $\mathcal{O}_F$-module, $\mathscr{F}_n := \mathfrak{F}/\varpi^n$ is a quasi-coherent sheaf, and $\mathfrak{F} = \lim_n \mathscr{F}_n$.*

A formal Banach sheaf is called *flat* if $\mathscr{F}_n$ is a flat $\mathscr{O}_{\mathfrak{X}}/\varpi^n$-module for all n. It is called *projective* if $\mathscr{F}_n$ is a projective $\mathscr{O}_{\mathfrak{X}}/\varpi^n$-module for all n. A formal Banach sheaf is called *small* if there exists a coherent sheaf $\mathscr{G}$ over $\mathfrak{X}$ with the property that $\mathscr{F}_1$ is the inductive limit of coherent sub sheaves $\mathscr{F}_1 = \mathrm{colim}_{j\in\mathbb{Z}_{\geq 0}}\mathscr{F}_{1,j}$ and $\mathscr{F}_{1,j}/\mathscr{F}_{1,j-1}$ is a direct summand of $\mathscr{G}$ for all $j \geq 0$.

The relevance of the smallness assumption is given by the following theorem:

Theorem 2.11. *Let $\mathfrak{X} \to \mathrm{Spf}\ \mathcal{O}_F$ be a finite type formal scheme and let $\mathfrak{F}$ be a small formal Banach sheaf. Assume that $\mathfrak{X}$ has an ample invertible sheaf, and that the generic fiber $\mathcal{X}$ of $\mathfrak{X}$ is affinoid. Then $\mathrm{H}^i(\mathfrak{X}, \mathfrak{F}) \otimes_{\mathcal{O}_F} F = 0$ for all $i > 0$.*

Proof. This is [AIP15], thm. A.1.2.2. In the reference, the formal scheme $\mathfrak{X}$ is assumed to be normal and quasi-projective, but the only property needed in the proof is the existence of an ample sheaf on $\mathfrak{X}$. □

Let $\mathfrak{X} \to \mathrm{Spf}\ \mathcal{O}_F$ be a finite type formal scheme, and let $\mathcal{X} \to \mathrm{Spa}(F, \mathcal{O}_F)$ be the generic fiber of $\mathfrak{X}$. Thus $(\mathcal{X}, \mathscr{O}^+_{\mathcal{X}}) = \lim_{\mathfrak{X}'}(\mathfrak{X}', \mathscr{O}_{\mathfrak{X}'})$ where the limit runs over all admissible blow-ups of $\mathfrak{X}$. Let $\mathfrak{F}$ be a Banach sheaf over $\mathfrak{X}$. For any admissible blow-up $f : \mathfrak{X}' \to \mathfrak{X}$, we let $\mathfrak{F}_{\mathfrak{X}'} = \lim_n f^\star\mathscr{F}_n$. We let $\mathscr{F} = \lim_{\mathfrak{X}'} \mathfrak{F}_{\mathfrak{X}'}[1/\varpi]$. This is a sheaf over $\mathcal{X}$ that we call the generic fiber of $\mathfrak{F}$.

Theorem 2.12. *Let $\mathfrak{X} \to \mathrm{Spf}\ \mathcal{O}_F$ be a finite type formal scheme, with generic fiber $\mathcal{X}$. We have the following properties:*

1. *There is a "generic fiber" functor going from the category of flat formal Banach sheaves over $\mathfrak{X}$ to the category of Banach sheaves over $\mathcal{X}$, described by the procedure $\mathfrak{F} \mapsto \mathscr{F}$.*
2. *If $\mathcal{U} \hookrightarrow \mathcal{X}$ is a quasi-compact open subset and $\mathfrak{U}' \hookrightarrow \mathfrak{X}'$ is a formal model for the map $\mathcal{U} \hookrightarrow \mathcal{X}$, $\mathscr{F}(\mathcal{U}) = \mathfrak{F}_{\mathfrak{X}'}(\mathfrak{U}')[1/\varpi]$.*
3. *The property (2) of definition 2.7 holds over the generic fiber of any affine covering of $\mathfrak{X}$.*

(4) *The generic fiber functor sends projective formal Banach sheaves to projective Banach sheaves.*
(5) *Let $\mathscr{F}$ be a Banach sheaf arising from a flat small formal Banach sheaf. Then for any affinoid $\mathcal{U} \hookrightarrow \mathcal{X}$ we have $\mathrm{H}^i(\mathcal{U}, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. The first three points are [AIP15], proposition A.2.2.3. We check the fourth point. Let $\mathfrak{U}$ be an open affine of $\mathfrak{X}$. Let $M_n = \mathrm{H}^0(\mathfrak{U}, M_n)$, $A_n = \mathrm{H}^0(\mathfrak{U}, \mathscr{O}_{\mathfrak{X}}/\varpi^n)$, $M = \lim_n M_n$ and $A = \lim_n A_n$. We claim that M is a direct factor of the completion of a free A-module. Let us pick a surjection $A_1^I \to M_1$ and for any n, we can lift it successively to surjections $A_n^I \to M_n$. We need to prove that we can find a compatible system of sections $s_n : M_n \to A_n^I$. It suffices to show that the map $\mathrm{Hom}_A(M_n, A_n^I) \to \mathrm{Hom}_A(M_{n-1}, A_{n-1}^I)$ is surjective. This follows from the short exact sequence $0 \to \mathrm{Hom}_{A_n}(M_n, A_1^I) \to \mathrm{Hom}_{A_n}(M_n, A_n^I) \to \mathrm{Hom}_{A_n}(M_n, A_{n-1}^I) \to 0$. We check the last point. Let $\mathfrak{X}$ be a formal model of $\mathcal{X}$ and $\mathfrak{F}$ be a small flat formal Banach sheaf over $\mathfrak{X}$. Let $\mathfrak{U}$ be an affine formal model of $\mathcal{U}$. Let $\cup_i \mathcal{U}_i = \mathcal{U}$ be a finite affinoid cover of $\mathcal{U}$. Let $\mathfrak{U}'$ be an admissible blow-up of $\mathfrak{U}$ with the property that $\cup_i \mathcal{U}_i = \mathcal{U}$ is the generic fiber of a covering of $\mathfrak{U}'$ and there is a map $\mathfrak{U}' \to \mathfrak{X}$ inducing the map $\mathcal{U} \to \mathcal{X}$. Note that $\mathfrak{U}'$ has an ample invertible sheaf, since it is a blow-up of an affine formal scheme. We can apply theorem 2.11 to $\mathfrak{F}_{\mathfrak{U}'}$, the pull-back to $\mathfrak{U}'$ of $\mathfrak{F}$ which is still small by flatness. This shows that the Čech cohomology of $\mathcal{U}$ with respect to the covering $\cup_i \mathcal{U}_i$ vanishes. Since this holds for any finite cover, and $\mathcal{U}$ is quasi-compact, we deduce that $\mathrm{H}^i(\mathcal{U}, \mathscr{F}) = 0$. □

Definition 2.13. *Let $\mathcal{X}$ be a finite type adic space over $\mathrm{Spa}(F, \mathcal{O}_F)$. A Banach sheaf $\mathscr{F}$ is called a small projective Banach sheaf if it arises as the generic fiber of a small projective formal Banach sheaf.*

Remark 2.14. A projective coherent sheaf over $\mathcal{X}$ is a small projective Banach sheaf by the flattening techniques of [RG71].

Remark 2.15. We don't know if, for $\mathcal{X} = \mathrm{Spa}(A, A^+)$ affinoid and a small projective Banach sheaf $\mathscr{F}$, it is true that $\mathscr{F}(\mathcal{X})$ is a projective Banach A-module and the map $\mathscr{F}(\mathcal{X}) \hat{\otimes}_A \mathscr{O}_{\mathcal{X}} \to \mathscr{F}$ is an isomorphism.

2.5.3. *Acyclicity of quasi-Stein spaces.* In our arguments, it will often be usefull to consider not only affinoid covers of adic spaces, but also some quasi-Stein covers.

Definition 2.16 ([Kie67], def. 2.3)**.** *We say that an adic space $\mathcal{X} \to \mathrm{Spa}(F, \mathcal{O}_F)$ is quasi-Stein if $\mathcal{X} = \cup_{i \in \mathbb{Z}_{\geq 0}} \mathcal{X}_i$ is a countable increasing union of finite type affinoid adic spaces $\mathcal{X}_i \to \mathrm{Spa}(F, \mathcal{O}_F)$ and $\mathscr{O}_{\mathcal{X}_{i+1}} \to \mathscr{O}_{\mathcal{X}_i}$ has dense image.*

Example 2.17. Here are some examples of quasi-Stein adic spaces:
- An affinoid space, like the unit ball $\mathbb{B}(0, 1)$.
- A Stein space like the open unit ball: $\mathbb{B}^o(0, 1) = \cup_n \mathbb{B}(0, |p^{\frac{1}{n}}|)$.
- A "mixed" situation like $\mathbb{B}^o(0, 1) \times_{\mathrm{Spa}(F, \mathcal{O}_F)} \mathbb{B}(0, 1)$.

We also recall the following classical acyclicity result:

Theorem 2.18 ([Kie67], Satz 2.4)**.** *Let $\mathcal{X}$ be a quasi-Stein adic space and let $\mathscr{F}$ be a coherent sheaf over $\mathcal{X}$. Then $\mathrm{H}^i(\mathcal{X}, \mathscr{F}) = 0$ for all $i > 0$.*

We also have:

Proposition 2.19. *Let $\mathcal{X} = \mathrm{Spa}(A, A^+)$ be an affinoid finite type adic space, let M be a projective Banach A-module and let $\mathscr{F} = M\hat{\otimes}_A \mathscr{O}_{\mathcal{X}}$. Let $\mathcal{U} \hookrightarrow \mathcal{X}$ be a quasi-Stein open subset. Then $\mathrm{H}^i(\mathcal{U}, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. We reduce to the case where M is orthonormalizable, and therefore to the case of $\mathscr{O}_{\mathcal{X}}$ where we can apply theorem 2.18. □

2.5.4. *Cohomology complexes.* We now illustrate how one can obtain complexes of Banach modules. We denote by $\mathcal{D}(F)$ the derived category of the category of F-vector spaces. We have natural functors: $\mathcal{K}^{proj}(F) \to \mathcal{D}(F)$ and $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))) \to \mathrm{Pro}_{\mathbb{N}}(\mathcal{D}(F)))$. Moreover, to any object in $\mathrm{Pro}_{\mathbb{N}}(\mathcal{D}(F)))$ we can attach its derived limit in $\mathcal{D}(F)$. In general derived categories, the derived limit (when it exists) is unique up to a non-unique quasi-isomorphism (see [Sta13], Tag 08 TB), but over a field there is no ambiguity. For an adic space $\mathcal{X}$ over $\mathrm{Spa}(F, \mathcal{O}_F)$, and a sheaf $\mathscr{F}$ of $\mathscr{O}_{\mathcal{X}}$-module, the cohomology groups $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F})$ are objects of the category $\mathcal{D}(F)$. Nevertheless, they often carry more structure and can be represented by complexes of Banach modules. We formalize this in this section.

Lemma 2.20. *Let $\mathcal{X}$ be a separated finite type adic space over $\mathrm{Spa}(F, \mathcal{O}_F)$. Let $\mathscr{F}$ be a projective Banach sheaf over $\mathcal{X}$. Let $\mathcal{U} \subseteq \mathcal{X}$ be a quasi-compact open subset. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed subset, with quasi-compact complement. Then one can naturally view $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathscr{F})$ as an object of $\mathcal{K}^{proj}(F)$.*

Proof. We have an exact triangle $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{U}, \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{U} \setminus (\mathcal{Z} \cap \mathcal{U}), \mathscr{F}) \stackrel{+1}{\to}$ and therefore, we are reduced to prove the claim for $\mathrm{R}\Gamma(\mathcal{U}, \mathscr{F})$ and $\mathrm{R}\Gamma(\mathcal{U} \setminus (\mathcal{Z} \cap \mathcal{U}), \mathscr{F})$. Finally, it suffices to prove that for a quasi-compact open $\mathcal{U} \subseteq \mathcal{X}$, $\mathrm{R}\Gamma(\mathcal{U}, \mathscr{F}) \in Ob(\mathcal{K}^{proj}(F))$. We can compute the cohomology by considering an affinoid covering of $\mathcal{U}$ for $\mathscr{F}$, and the associated Čech complex by [Hub94], thm. 2.5. Then each of the terms of the Čech complex carries a canonical structure of Banach K-algebra. Any two complexes obtained that way are quasi-isomorphic, hence homotopic. □

Lemma 2.21. *Let $\mathcal{X}$ be a separated finite type adic space over $\mathrm{Spa}(F, \mathcal{O}_F)$. Let $\mathscr{F}$ be a projective Banach sheaf over $\mathcal{X}$. Let $\mathcal{U} \subseteq \mathcal{X}$ be an open subset which is a finite union of quasi-Stein spaces. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed subset, whose complement is a finite union of quasi-Stein spaces. Then one can naturally view $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathscr{F})$ as an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F)))$.*

Proof. As before, we are reduced to see that for an open $\mathcal{U} \subseteq \mathcal{X}$ which is a finite union of quasi-Stein spaces, $\mathrm{R}\Gamma(\mathcal{U}, \mathscr{F}) \in Ob(\mathrm{Pro}_{\mathbb{N}}\mathcal{K}^{proj}(F))$. We let $\mathcal{U} = \cup_k \mathcal{V}_k$ be a finite covering of $\mathcal{U}$ by acyclic quasi-Stein spaces $\mathcal{V}_k$. We let $\mathcal{V}_k = \cup_{i\geq 0}\mathcal{V}_{k,i}$ where $\mathcal{V}_{k,i}$ is affinoid. We let $\mathcal{U}_i = \cup_k \mathcal{V}_{k,i}$. We let $\mathrm{R}\Gamma(\mathcal{U}, \mathscr{F}) = \text{“}\lim\text{”}\mathrm{R}\Gamma(\mathcal{U}_i, \mathscr{F})$. We can prove that this is independent of the covering $\mathcal{U} = \cup_{k,i}\mathcal{V}_{k,i}$. Indeed, let $\mathcal{U} = \cup_{k',i}\mathcal{W}_{k',i}$ be another covering. By considering intersections, we may assume that $\cup_{k',i}\mathcal{W}_{k',i}$ refines $\cup_{k,i}\mathcal{V}_{k,i}$. If we let $\mathcal{U}'_i = \cup_{k'}\mathcal{W}_{k',i}$, we see that for all i, there exists $i' \geq i \geq i''$ such that $\mathcal{U}'_{i''} \subseteq \mathcal{U}_i \subseteq \mathcal{U}'_{i'}$. Therefore, the limits are equal. □

2.5.5. *Compact morphisms in the cohomology of adic spaces.* We give some examples of compact morphisms arising from maps between the cohomology of adic spaces. First, let us fix a standard notation. Let T be a topological space and let S

be a subset of T. Then we denote by $\overline{S}$ the closure of S in T and by $\overset{\circ}{S}$ the interior of S in T.

Lemma 2.22. *Let $\mathcal{X} \to \mathrm{Spa}(F, \mathcal{O}_F)$ be a proper adic space and let $\mathscr{F}$ be a locally free sheaf of finite rank over $\mathcal{X}$. Let $\mathcal{U}' \subseteq \mathcal{U} \subseteq \mathcal{X}$ be quasi-compact open subsets. Assume that $\overline{\mathcal{U}'} \subseteq \mathcal{U}$. Then the map*

$$\mathrm{R}\Gamma(\mathcal{U}, \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{U}', \mathscr{F})$$

is compact.

Proof. We claim that there exists a formal model $\mathfrak{X} \to \mathrm{Spf}\ \mathcal{O}_F$ of $\mathcal{X}$, and two opens $\mathfrak{U}$ and $\mathfrak{U}'$ of $\mathfrak{X}$ with generic fibers $\mathcal{U}$ and $\mathcal{U}'$, such that $\overline{\mathfrak{U}'} \subseteq \mathfrak{U}'$. Remark that $\overline{\mathfrak{U}'} \to \mathrm{Spf}\ \mathcal{O}_F$ is proper because $\mathcal{X}$ was assumed to be proper ([L$\ddot{9}$0], Th. 3.1). We deduce that $\mathcal{U}'$ is relatively compact in $\mathcal{U}$ ([L$\ddot{9}$0], lem. 2.5). We prove the claim. Recall that the ringed space $(\mathcal{X}, \mathscr{O}^+_{\mathcal{X}})$ is the inverse limit of the ringed spaces $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ where $\mathfrak{X}$ runs over all the formal models of $\mathcal{X}$. For a cofinal subset of $\mathfrak{X}$, we have opens $\mathfrak{U}_{\mathfrak{X}}$ and $\mathfrak{U}'_{\mathfrak{X}}$ of $\mathfrak{X}$ with generic fiber $\mathcal{U}$ and $\mathcal{U}'$. We let $\overline{\mathcal{U}'}_{\mathfrak{X}}$ be the generic fiber of $\overline{\mathfrak{U}'}_{\mathfrak{X}}$. Then $\overline{\mathcal{U}'} = \cap_{\mathfrak{X}} \overline{\mathcal{U}'}_{\mathfrak{X}}$. The topological space $\mathcal{X}$ equipped with the constructible topology is compact (in fact this is a profinite set). We have that $(\mathcal{U})^c = \cup_{\mathfrak{X}} (\overline{\mathcal{U}'}_{\mathfrak{X}})^c \cap (\mathcal{U})^c$. Since $(\mathcal{U})^c$ is compact, we deduce that there is a model $\mathfrak{X}$ such that $\overline{\mathcal{U}'}_{\mathfrak{X}} \subseteq \mathcal{U}$, and therefore $\overline{\mathfrak{U}'}_{\mathfrak{X}} \subseteq \mathfrak{U}_{\mathfrak{X}}$. This finishes the proof of the claim.

Let $\mathcal{U}' = \cup_{i \in I} \mathcal{U}'_i$ be a finite affinoid cover of $\mathcal{U}'$. By [L$\ddot{9}$0], thm. 5.1, for each i, there exists an affinoid $\mathcal{U}'_i \subseteq \mathcal{U}_i \subseteq \mathcal{U}$ such that $\mathcal{U}'_i$ is relatively compact in $\mathcal{U}_i$ (equivalently $\overline{\mathcal{U}'_i} \subseteq \mathcal{U}_i$). Let $\mathcal{U}'' = \cup_{i \in I} \mathcal{U}_i$. We claim that the map

$$\mathrm{R}\Gamma(\mathcal{U}'', \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{U}', \mathscr{F})$$

is compact. Indeed, these cohomology can be represented by the Čech complex with respect to $\cup_i \mathcal{U}_i$ and $\cup_i \mathcal{U}'_i$. By the same argument as in [KL05], prop. 2.4.1, we find that the maps $\mathscr{F}(\cap_{J \subseteq I} \mathcal{U}_i) \to \mathscr{F}(\cap_{J \subseteq I} \mathcal{U}'_i)$ are compact. Finally, the map of the lemma is compact because it factors over a compact map. ☐

Lemma 2.23. *Let $\mathcal{X} \to \mathrm{Spa}(F, \mathcal{O}_F)$ be a proper adic space and let $\mathscr{F}$ be a locally free sheaf of finite rank over $\mathcal{X}$. Let $\mathcal{U}' \subseteq \mathcal{U}$ be two quasi-compact open, and $\mathcal{Z} \subseteq \mathcal{Z}'$ be two closed subspaces, with quasi-compact complements. Assume that $\overline{\mathcal{U}}' \subseteq \mathcal{U}$ and $\mathcal{Z} \subseteq \overset{\circ}{\mathcal{Z}}{}'$. Then the map*

$$\mathrm{R}\Gamma_{\mathcal{Z} \cap \mathcal{U}}(\mathcal{U}, \mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}' \cap \mathcal{U}'}(\mathcal{U}', \mathscr{F})$$

is compact.

Proof. It suffices to see that $\mathrm{R}\Gamma(\mathcal{U}, \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{U}', \mathscr{F})$ and $\mathrm{R}\Gamma(\mathcal{U} \setminus (\mathcal{Z} \cap \mathcal{U}), \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{U}' \setminus (\mathcal{Z}' \cap \mathcal{U}'), \mathscr{F})$ are compact. This follows from the fact that $\overline{\mathcal{U}}' \subseteq \mathcal{U}$ and $\overline{\mathcal{U}' \setminus (\mathcal{Z}' \cap \mathcal{U}')} \subseteq \overline{\mathcal{U}}' \setminus (\overset{\circ}{\mathcal{Z}}{}' \cap \overline{\mathcal{U}}') \subseteq \mathcal{U} \setminus (\mathcal{Z} \cap \mathcal{U})$ and the previous lemma. ☐

We now give a stronger form of the lemma.

Lemma 2.24. *Assume that $\mathcal{X}$ is proper. Let $\mathcal{U}' \subseteq \mathcal{U}$ be two open subsets which are finite unions of quasi-Stein spaces, and $\mathcal{Z} \subseteq \mathcal{Z}'$ be two closed subspaces, whose complements are finite unions of quasi-Stein spaces. Assume that there exists a quasi-compact open $\mathcal{U}''$ such that $\mathcal{U}' \cap \mathcal{Z}' \subseteq \mathcal{U}''$ with $\overline{\mathcal{U}''} \subseteq \mathcal{U}$, as well as two closed* *subset* $\mathcal{Z}'' \subseteq \mathcal{Z}'''$ *with quasi-compact complement, such that* $\mathcal{Z}\cap\mathcal{U} \subseteq \mathcal{Z}^{''}$, $\mathcal{Z}^{''} \subseteq \mathring{\mathcal{Z}}'''$ *and* $\mathcal{Z}^{'''} \subseteq \mathcal{Z}'$. *Then the map*

$$\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F})$$

is compact in $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$.

Proof. We can write $\mathcal{U} = \cup_n \mathcal{U}_n$ with $\mathcal{U}_n$ quasi-compact. Since $\overline{\mathcal{U}}'' \subseteq \mathcal{U}$ and $\mathcal{X}$ equipped with the constructible topology is compact, we deduce that $\overline{\mathcal{U}}'' \subseteq \mathcal{U}_n$ for n large enough. The map $\mathrm{R}\Gamma_{\mathcal{Z}''\cap\mathcal{U}_n}(\mathcal{U}_n,\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}'''\cap\mathcal{U}''}(\mathcal{U}'',\mathscr{F})$ is compact by lemma 2.23. There is a restriction-corestriction map $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}''\cap\mathcal{U}_n}(\mathcal{U}_n,\mathscr{F})$.

We also have a restriction-corestriction map

$$\mathrm{R}\Gamma_{\mathcal{Z}'''\cap\mathcal{U}''}(\mathcal{U}'',\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}'\cap\mathcal{U}'',\mathscr{F}).$$

On the other hand, $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F}) = \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}'\cap\mathcal{U}'',\mathscr{F})$. All together, we deduce that the map

$$\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F})$$

factors through the compact map $\mathrm{R}\Gamma_{\mathcal{Z}''\cap\mathcal{U}_n}(\mathcal{U}_n,\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}'''\cap\mathcal{U}''}(\mathcal{U}'',\mathscr{F})$ and is compact. □

We finally conclude this section with a last lemma where we deal with Banach sheaves which are not necessarily coherent.

Lemma 2.25. *Let* $\mathcal{X}, \mathcal{U}, \mathcal{U}', \mathcal{Z}, \mathcal{Z}'$ *be as in lemma 2.24. Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be two projective Banach sheaves and let* $\phi : \mathscr{F} \to \mathscr{G}$ *be a compact morphism. The morphism*

$$\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{G})$$

is compact in $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$.

Proof. Easy and left to the reader. □

2.6. **Integral structures on Banach sheaves.** We now consider integral structures on Banach sheaves, but this time more in the spirit of analytic geometry. We let $\mathcal{X}$ be a separated adic space locally of finite type over $\mathrm{Spa}(F,\mathcal{O}_F)$. We let $\mathscr{F}$ be a projective Banach sheaf over $\mathcal{X}$. We can view $\mathscr{F}$ as a sheaf on the étale site of $\mathcal{X}$ by [BG98].

Definition 2.26. *An integral structure on* $\mathscr{F}$ *is a sheaf* $\mathscr{F}^+$ *of* $\mathscr{O}^+_{\mathcal{X}}$*-modules on the étale site of* X*, such that:*

1. $\mathscr{F}^+ \hookrightarrow \mathscr{F}$ *and* $\mathscr{F}^+ \otimes_{\mathcal{O}_F} K = \mathscr{F}$*,*
2. *There is an étale cover* $\coprod U_i \to \mathcal{X}$ *by affinoid spaces such that* $\mathscr{F}^+(U_i)$ *is the completion of a free* $\mathscr{O}^+_{X}(U_i)$*-module of finite rank and the canonical map* $\mathscr{F}^+(U_i)\hat{\otimes}_{\mathscr{O}^+_{\mathcal{X}}(U_i)}\mathscr{O}^+_{U_i} \to \mathscr{F}^+|_{U_i}$ *is an isomorphism.*

Remark 2.27. A stronger property would be to ask that the étale cover $\coprod U_i \to \mathcal{X}$ is in fact an analytic cover. In our applications, this stronger property will not be satisfied. Indeed, we will produce sheaves arising from torsors under various groups, and these torsors are usually only trivial locally for the étale topology. However, we will not consider the étale cohomology $\mathrm{H}^i_{et}(\mathcal{X},\mathscr{F}^+)$, but only the analytic cohomology $\mathrm{H}^i_{an}(\mathcal{X},\mathscr{F}^+)$.

Lemma 2.28. *Assume that $\mathcal{X}$ is reduced. Let $U \to \mathcal{X}$ be an étale map. Then $\mathscr{F}^+(U)$ is an open and bounded submodule of $\mathscr{F}(U)$.*

Proof. Let i be a finite set and let $\coprod_{i\in I} U_i \to U$ be an étale cover such each U_i is affinoid and $\mathscr{F}^+|_{U_i}$ is the completion of a free sheaf of $\mathscr{O}^+_{U_i}$-modules. By the sheaf property we have an exact sequence:

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \to \prod_{i,j} \mathscr{F}(U_i \times_U U_j)$$

and $\mathscr{F}(U)$ is a closed Banach subspace of $\prod_i \mathscr{F}(U_i)$. Since $\prod_i \mathscr{F}^+(U_i)$ is open and bounded in $\prod_i \mathscr{F}(U_i)$ (using that $\mathscr{O}^+_{\mathcal{X}}(U_i)$ is bounded by the reduced hypothesis), $\mathscr{F}^+(U) = \prod_i \mathscr{F}^+(U_i) \cap \mathscr{F}(U)$ is open and bounded in $\mathscr{F}^+(U)$. □

We now will elaborate on a result of Bartenwerfer which we first recall.

Theorem 2.29 ([Bar78])**.** *Let $\mathcal{X}$ be an affinoid smooth adic space over* $\mathrm{Spa}(F, \mathcal{O}_F)$*. There exists $N \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{H}^i_{an}(\mathcal{X}, \mathscr{O}^+_{\mathcal{X}})$ is annihilated by p^N for all $i > 0$.*

Proof. Bartenwerfer's result is stated for Chech cohomology. By [Pil20] prop. 3.1.1, this implies the claim for cohomology. □

Lemma 2.30. *Let $\mathcal{X}$ be an affinoid smooth adic space over* $\mathrm{Spa}(F, \mathcal{O}_F)$*. Let $\mathscr{F}$ be a projective Banach sheaf which is assumed to be associated to its global section (i.e. satisfies point* (2) *in definition 2.7). Let $\mathscr{F}^+$ be an integral structure on $\mathscr{F}$. There exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $i > 0$ the cohomology groups $\mathrm{H}^i_{an}(X, \mathscr{F}^+)$ are annihilated by p^N.*

Proof. Let $\mathcal{X} = \mathrm{Spa}(A, A^+)$. By assumption, $\mathscr{F} = M\hat{\otimes}_A \mathscr{O}_{\mathcal{X}}$ is associated to a projective Banach A-module M. Let I be a set such that $A(I) = M \oplus N$. Let $M^+ = A^+(I) \cap M$ and $N^+ = A^+(I) \cap N$. The injective map $M^+ \oplus N^+ \to A^+(I)$ has cokernel of bounded torsion. Moreover, if we let M_+ be the image of $A^+(I)$ in M under the projection orthogonal to N, then $M^+ \hookrightarrow M_+$ has cokernel of bounded torsion. We deduce that there exists an integer N such that the multiplication by p^N maps $M^+ \to M^+$ factors through: $M^+ \to A^+(I) \to M^+$ (where the first map is the inclusion, the second map is the orthogonal projection with respect to N composed with multiplication by p^N and followed by the inclusion $p^N M_+ \subseteq M^+$). It follows from theorem 2.29 that $\mathrm{H}^i(\mathcal{X}, \mathscr{O}^+_{\mathcal{X}}\hat{\otimes}_{A^+} A^+(I))$ is of bounded torsion for all $i > 0$. We let $\mathscr{M}^+$ be the subsheaf of $\mathscr{O}^+_{\mathcal{X}}\hat{\otimes}_{A^+} A^+(I) \cap \mathscr{F}$ equal to the image of the sheaf associated to the presheaf $\mathscr{O}^+_{\mathcal{X}}\hat{\otimes}_{A^+} M^+$ (in other words, it is the subsheaf of $\mathscr{F}$ of sections which can locally be written as tensors in $\mathscr{O}^+_{\mathcal{X}}\hat{\otimes}_{A^+} M^+$).

We see that multiplication by $p^N : \mathscr{M}^+ \to \mathscr{M}^+$ factors through $\mathscr{M}^+ \to \mathscr{O}^+_{\mathcal{X}}\hat{\otimes}_{A^+} A^+(I) \to \mathscr{M}^+$. We deduce that $\mathrm{H}^i(\mathcal{X}, \mathscr{M}^+)$ is of bounded torsion for all $i > 0$. After rescaling, we may assume that $M^+ \subseteq \mathscr{F}^+(\mathcal{X})$, with cokernel of bounded torsion by lemma 2.28. We therefore get a morphism: $\mathscr{M}^+ \to \mathscr{F}^+$. We claim that this morphism has cokernel of bounded torsion. Let $\cup_{i\in I} U_i \to \mathcal{X}$ be an affinoid étale covering with the property that $\mathscr{F}^+|_{U_i}$ is associated with its global sections. We may assume that the set I is finite. It suffices to show that $\mathscr{O}^+_{\mathcal{X}}(U_i)\hat{\otimes}_{A^+} M^+ \to \mathscr{F}^+(U_i)$ has cokernel of bounded torsion. This follows since the image of $\mathscr{O}^+_{\mathcal{X}}(U_i)\hat{\otimes}_{A^+} M^+$ is open. We finally deduce that $\mathrm{H}^i_{an}(X, \mathscr{F}^+)$ is of bounded torsion for all $i > 0$. □

2.7. **Duality for analytic adic spaces.** In this section we fix $\mathcal{X}$ a proper smooth adic space over $\mathrm{Spa}(F, \mathcal{O}_F)$ of pure dimension d. Let $\Omega^d_{\mathcal{X}/F}$ be the canonical sheaf, equal to $\Lambda^d\Omega^1_{\mathcal{X}/F}$. Let $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed subset, equal to the closure of a quasi-compact open subset $\mathcal{U}$ of $\mathcal{X}$. One can consider the cohomology group $\mathrm{H}^d_{\mathcal{Z}}(\mathcal{X}, \Omega^d_{\mathcal{X}/F})$.

Remark 2.31. We give some translation to the language of [GK00]. Let $\mathcal{U}^\dagger$ be the dagger space attached to $\mathcal{U} \hookrightarrow \mathcal{X}$. And let $\mathscr{F}$ be a coherent sheaf defined on a neighborhood of $\mathcal{Z}$ in $\mathcal{X}$. Then $\mathrm{H}^i(\mathcal{U}^\dagger, \mathscr{F}) = \mathrm{H}^i(\mathcal{Z}, \iota^{-1}\mathscr{F})$ and $\mathrm{H}^d_c(\mathcal{U}^\dagger, \mathscr{F}) = \mathrm{H}^i_{\mathcal{Z}}(\mathcal{X}, \mathscr{F})$.

Theorem 2.32 ([GK00], [Bey97]).
1. *There is a trace map* $\mathrm{tr}_{\mathcal{Z}} : \mathrm{H}^d_{\mathcal{Z}}(\mathcal{X}, \Omega^d_{\mathcal{X}/F}) \to F$.
2. *If* $\mathcal{Z} \subseteq \mathcal{Z}'$, *there is a factorization*
$$\mathrm{tr}_{\mathcal{Z}} : \mathrm{H}^d_{\mathcal{Z}}(\mathcal{X}, \Omega^d_{\mathcal{X}/K}) \to \mathrm{H}^d_{\mathcal{Z}'}(\mathcal{X}, \Omega^d_{\mathcal{X}/F}) \overset{\mathrm{tr}_{\mathcal{Z}'}}{\to} F.$$
3. *For any coherent sheaf* $\mathscr{F}$ *defined on a neighborhood of* $\mathcal{Z}$ *the map* $\mathrm{tr}_{\mathcal{Z}}$ *induces a pairing:*
$$\mathrm{Ext}^i_{\iota^{-1}\mathscr{O}_{\mathcal{X}}}(\iota^{-1}\mathscr{F}, \iota^{-1}\Omega^d_{\mathcal{X}/F}) \times \mathrm{H}^{d-i}_{\mathcal{Z}}(\mathcal{X}, \mathscr{F}) \to K$$
4. *When* $\mathcal{Z} = \mathcal{X}$ *the cohomology groups are finite dimensional* K*-vector spaces, and the pairing is perfect.*
5. *When* $\mathcal{U}$ *is affinoid and* $\mathscr{F}$ *is a locally free sheaf,* $\mathrm{Ext}^0_{\iota^{-1}\mathscr{O}_{\mathcal{X}}}(\iota^{-1}\mathscr{F}, \iota^{-1}\Omega^d_{\mathcal{X}/K})$ *is a compact inductive limit of* F*-Banach spaces,* $\mathrm{H}^d_{\mathcal{Z}}(\mathcal{X}, \mathscr{F})$ *is a compact projective limit of* F*-Banach spaces, and the pairing is a topological duality between locally convex* F*-Banach spaces. Moreover,* $\mathrm{Ext}^i_{\iota^{-1}\mathscr{O}_{\mathcal{X}}}(\iota^{-1}\mathscr{F}, \iota^{-1}\Omega^d_{\mathcal{X}/K})$ *and* $\mathrm{H}^{d-i}_{\mathcal{Z}}(\mathcal{X}, \mathscr{F})$ *vanish for* $i \neq 0$.

3. Flag variety

3.1. **Bruhat decomposition.** Let F be a non archimedean local field of mixed characteristic with residue field k of characteristic p, discrete valuation v and uniformizer ϖ. Let $G \to \mathcal{O}_F$ be a split reductive group with maximal torus T contained in a borel B with unipotent radical U. Let $B \subseteq P$ be a parabolic. We choose a Levi $M \subseteq P$.

Remark 3.1. In our application to Shimura varieties, the unipotent radical of P will be abelian. Nevertheless, this assumption is not relevant for the moment.

We let Φ be the set of roots, Φ^+ be the subset of positive roots corresponding to our choice of B and $\Phi^- = -\Phi^+$. We let Φ^+_M be the subset of positive roots which lie in the Lie algebra of M and $\Phi^{+,M} = \Phi^+ \setminus \Phi^+_M$. We let $\Phi^-_M = -\Phi^+_M$ and $\Phi^{-,M} = -\Phi^{+,M}$.

We let W be the Weyl group of G. We denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function. For each $w \in W$, we choose a representative in $N(T)$ that we still denote w. The group W acts on the left on the cocharacter group $X_\star(T)$ and on the character group $X^\star(T)$ on the left as well via the formula $w\kappa(t) = \kappa(w^{-1}tw)$ for $\kappa \in X^\star(T)$ and $w \in W$. Let W_M be the Weyl group of M. The quotient $W_M\backslash W$ has a set of coset representatives of minimal length (the Kostant representatives) called ${}^MW$. This is the subset of W of elements w that verify $\Phi^+_M \subseteq w\Phi^+$.

Let $FL = P\backslash G \to \mathrm{Spec}\ \mathcal{O}_F$ be the flag variety associated with P. The group G acts on the right on FL.

For any $w \in {}^MW$, we let $C_w = P\backslash PwB$ be the Bruhat cell corresponding to w. We have the decomposition into B-orbits $FL = \coprod_{w\in {}^MW} C_w$. We can also consider the opposite Bruhat cell: $C^w = P\backslash Pw\overline{B}$ for the opposite Borel $\overline{B}$.

We let X_w be the Schubert variety equal to the Zariski closure of C_w in FL. We also let X^w be the opposite Schubert variety, equal to the Zariski closure of C^w in FL. There is a partial order $\leq$ on ${}^MW$ for which $X_w = \cup_{w'\leq w} C_{w'}$ and $X^w = \cup_{w'\geq w} C^{w'}$. For the length function $\ell : {}^MW \to [0, \mathrm{dim} FL]$, we have $\ell(w) = \dim C_w$ (here dimensions are relative dimensions over Spec $\mathcal{O}_F$).

We also define for all $w \in {}^MW$, $Y_w = \cup_{w'\geq w} C_w$. This is an open subscheme of FL containing C_w.

Lemma 3.2. *We have an inclusion $X^w \hookrightarrow Y_w$.*

Proof. By [BL03], I, lemma 1, we know that $X^w \cap X_v \neq \emptyset \Leftrightarrow v \geq w$. We deduce that $X^w \cap C_v \neq \emptyset \Rightarrow v \geq w$, so that $X^w \subseteq \cup_{v\geq w} C_v$. □

The following lemma gives a description of the Bruhat cells. For all $\alpha \in \Phi$, we let U_α be the one parameter subgroup corresponding to α.

Lemma 3.3. *The product map (for any ordering of the factors)*

$$\begin{array}{rcl} \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap \Phi^+} U_\alpha & \to & C_w \\ (x_\alpha) & \mapsto & w\prod_\alpha x_\alpha \end{array}$$

is an isomorphism of schemes.

Proof. We need to prove that the map $\prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^+} U_\alpha \to (B \cap w^{-1}Pw)\backslash B$ is an isomorphism. This follows easily from the following facts:

- $B = T \times \prod_{\alpha\in\Phi^+} U_\alpha$ (in any order),
- $\Phi^+ = (w^{-1}\Phi^{-,M}) \cap \Phi^+ \coprod (w^{-1}\Phi^+) \cap \Phi^+ \coprod (w^{-1}\Phi^-_M) \cap \Phi^+$,
- $B \cap w^{-1}Pw = T \times \prod_{\alpha\in(w^{-1}\Phi^+)\cap\Phi^+ \coprod (w^{-1}\Phi^-_M)\cap\Phi^+} U_\alpha$.

□

We also introduce the open subset U_w, which contains C_w, defined to be the image of the map (for any ordering of the roots):

$$\begin{array}{rcl} \prod_{\alpha \in w^{-1}\Phi^{-,M}} U_\alpha & \to & FL \\ (x_\alpha) & \mapsto & w\prod_\alpha x_\alpha. \end{array}$$

This is the right the translate of the big cell C_{w_0} by $w_0^{-1}w$.

3.2. **Interlude: the cohomology of the flag variety.** In this subsection, which is independent of the rest of the paper, we discuss the coherent cohomology of the flag variety following [Kem78], section 12. We assume here that $P = B$ is a Borel subgroup. We consider the stratification $Z_0 = FL \supseteq Z_1 \supseteq \cdots \supseteq Z_d \supseteq Z_{d+1} = \emptyset$ where $d = \mathrm{dim} FL$ and $Z_i = \bigcup_{w\in W, \ell(w)=d-i} X_w$. Let $\kappa \in X^\star(T)$. We associate to κ

a G-equivariant line bundle $\mathcal{L}_\kappa$ on FL. If $\pi : G \to FL$ is the projection map, then for any open $U \hookrightarrow FL$,

$$\mathcal{L}_\kappa(U) = \{f : \pi^{-1}(U) \to \mathbb{A}^1 \mid f(bu) = w_0\kappa(b)f(u)\}.$$

We consider the spectral sequence

$$\mathrm{H}^{p+q}_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) \Rightarrow \mathrm{H}^{p+q}(FL, \mathcal{L}_\kappa).$$

In order to study this spectral sequence, we need the following basic result:

Lemma 3.4. *Let $m \geq n$. Let $\mathbb{A}^m = \mathrm{Spec}\, \mathcal{O}_F[X_1, \cdots, X_m]$ and $\mathbb{A}^n = \mathrm{Spec}\, \mathcal{O}_F[X_1, \cdots, X_n]$. Let $\mathbb{A}^n \hookrightarrow \mathbb{A}^m$ be the closed immersion given by $X_{n+1} = \cdots = X_m = 0$. We have $\mathrm{H}^i_{\mathbb{A}^n}(\mathbb{A}^m) = 0$ if $i \neq m-n$, and*

$$\mathrm{H}^{m-n}_{\mathbb{A}^n}(\mathbb{A}^m) = \bigoplus_{k_1, \cdots, k_n \geq 0,\ k_{n+1}, \cdots, k_m < 0} \mathcal{O}_F \prod_{i=1}^m X_i^{k_i}$$

Proof. One can use a Koszul complex. See [Gro05], exposé II, proposition 5. □

Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. We define the dotted action of W on $X^\star(T)$ by $w \cdot \kappa = w(\kappa + \rho) - \rho$. We also define $\mathrm{H}^\star_w(FL, \mathcal{L}_\kappa) = \mathrm{H}^\star_{C_w}(U_w, \mathcal{L}_\kappa)$.

Lemma 3.5. *We have a decomposition*

$$\mathrm{H}^{p+q}_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) = \bigoplus_{w \in W, \ell(w) = d-p} \mathrm{H}^{p+q}_w(FL, \mathcal{L}_\kappa)$$

and these groups vanish when $q \neq 0$. Moreover, the character of T acting on $\mathrm{H}^{d-\ell(w)}_w(FL, \mathcal{L}_\kappa)$ is given by the formula:

$$\frac{(w^{-1}w_0) \cdot \kappa}{\prod_{\alpha \in \Phi^-}(1-\alpha)}$$

Proof. Since $Z_p \setminus Z_{p+1} = \bigcup_{w \in W, \ell(w) = d-p} C_w$ and U_w is a neighborhood of C_w in FL, we deduce from lemma 2.1 that

$$\mathrm{H}^{p+q}_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) = \oplus_{w \in W, \ell(w) = d-p} \mathrm{H}^{p+q}_{C_w}(U_w, \mathcal{L}_\kappa).$$

We have $U_w \simeq w\mathbb{A}^d$, with coordinates X_α, $\alpha \in w^{-1}\Phi^-$. We have $C_w \simeq w\mathbb{A}^{\ell(w)}$ with coordinates X_α, $\alpha \in w^{-1}\Phi^- \cap \Phi^+$. Moreover, for $t \in T$ we have the formula $w \prod X_\alpha t = wtw^{-1}w \prod \mathrm{Ad}(t^{-1})(X_\alpha)$. In particular, we deduce easily that

$$\mathrm{H}^{p+q}_{C_w}(U_w, \mathcal{L}_\kappa) = \mathrm{H}^{p+q}_{C_w}(U_w, \mathscr{O}_{FL}) \otimes \mathcal{O}_F(w^{-1}w_0\kappa).$$

We deduce from lemma 3.4, that $\mathrm{H}^{p+q}_{C_w}(U_w, \mathcal{L}_\kappa)$ is concentrated in degree p, and the cohomology is isomorphic to the free $\mathcal{O}_F$-module:

$$\bigoplus_{k_\alpha \geq 0\ \forall \alpha \in w^{-1}\Phi^- \cap \Phi^+, k_\alpha < 0\ \forall \alpha \in w^{-1}\Phi^- \cap \Phi^-} \mathcal{O}_F \prod X_\alpha^{k_\alpha}$$

We can compute the character of the T-action. We have

$$\begin{aligned} \mathrm{ch}\big(\mathrm{H}^p_{C_w}(U_w, \mathscr{O}_{FL})\big) &= \sum_{k_\alpha \geq 0\ \forall \alpha \in w^{-1}\Phi^+ \cap \Phi^-, k_\alpha > 0\ \forall \alpha \in w^{-1}\Phi^- \cap \Phi^-} \prod \alpha^{k_\alpha} \\ &= \frac{\prod_{\alpha \in w^{-1}\Phi^- \cap \Phi^-} \alpha}{\prod_{\alpha \in \Phi^-}(1-\alpha)} \\ &= \frac{\rho^{-1}.w^{-1}\rho^{-1}}{\prod_{\alpha \in \Phi^+}(1-\alpha)} \end{aligned}$$

It follows that

$$\mathrm{ch}\big(\mathrm{H}^p_{C_w}(U_w,\mathcal{L}_\kappa)\big) = w^{-1}w_0\kappa.\rho^{-1}.w^{-1}\rho^{-1}\frac{1}{\prod_{\alpha\in\Phi^+}(1-\alpha)}$$

We conclude by noting that $w^{-1}w_0\kappa.\rho^{-1}.w^{-1}\rho^{-1} = (w^{-1}w_0)\cdot\kappa$. □

It follows from the lemma that the following complex (the Grothendieck-Cousin complex):

$$\mathcal{C}ous(\kappa) : 0 \to \mathrm{H}^0_{Z_0/Z_1}(FL,\mathcal{L}_\kappa) \to \cdots \to \mathrm{H}^d_{Z_d/Z_{d+1}}(FL,\mathcal{L}_\kappa) \to 0$$

computes $\mathrm{R}\Gamma(FL,\mathcal{L}_\kappa)$. Each group decomposes

$$\mathrm{H}^p_{Z_p/Z_{p+1}}(FL,\mathcal{L}_\kappa) = \bigoplus_{w\in W,\ell(w)=d-p}\mathrm{H}^p_w(FL,\mathcal{L}_\kappa)$$

and we have established a precise formula for the weights of T on each module.

There is a partial order on $X^\star(T)$ where $\lambda \geq 0$ if and only if λ is a sum of positive roots. Lemma 3.5 tells us that the weights occurring in $\mathrm{H}^{d-l(w)}_w(FL,\mathcal{L}_\kappa)$ are exactly those which are $\leq (w^{-1}w_0)\cdot\kappa$. In particular certain "big weights" will occur in as few of the terms of the Cousin complex as possible. We begin with the following lemma:

Lemma 3.6. *Let $\nu \in X^\star(T)$ be such that $\nu+\rho$ is dominant. Then the following conditions on a weight $\lambda \in X^\star(T)$ are equivalent:*

1. *$\lambda \not\leq w\cdot\nu$ for all $w\in W$ with $w\cdot\nu\neq\nu$.*
2. *$\lambda \not\leq s_\alpha\cdot\nu$ for all $\alpha\in\Delta$ with $s_\alpha\cdot\nu\neq\nu$.*

Moreover if we additionally assume that $\lambda\leq\nu$ then we have the further equivalent condition:

3. *$\lambda = \nu - \sum_{\alpha\in\Delta} n_\alpha\alpha$ with $n_\alpha < \langle\alpha^\vee,\nu\rangle+1$ for all $\alpha\in\Delta$ with $\langle\alpha^\vee,\nu\rangle+1>0$.*

Proof. Clearly the first condition implies the second. For the converse, writing w as a reduced product of simple reflections, there must be at least one factor s_α with $s_\alpha\cdot\nu\neq\nu$. Then $w\geq s_\alpha$, and so by lemma 5.55, $w\cdot\nu\leq s_\alpha\cdot\nu$, and hence $\lambda\not\leq s_\alpha\cdot\nu$ implies $\lambda\not\leq w\cdot\nu$.

The equivalence of the second and third points follows from the formula $s_\alpha\cdot\nu = \nu-(\langle\alpha^\vee,\nu\rangle+1)\alpha$. □

We say that a weight λ satisfying the conditions of the proposition has big weight (with respect to μ) and for a T-module M which is a direct sum of its weight spaces we denote by $M^{bw(\mu)}$ the direct sum of its weight spaces corresponding to big weights. We note that if $\nu+\rho$ is regular the last condition may be expressed as $\lambda > \nu - \sum_{\alpha\in\Delta}\langle(\alpha^\vee,\nu\rangle+1)\alpha$.

Let $C(\kappa) = \{w\in W \mid (w^{-1}w_0)\cdot\kappa \in X^\star(T)^+_{\mathbb{Q}}-\rho\}$. This set is nonempty, and $\kappa+\rho$ is regular if and only if it consists of a single element. For example, if κ is dominant, $C(\kappa)=\{w_0\}$. We let $\nu=(w^{-1}w_0)\cdot\kappa$ for $w\in C(\kappa)$ (this is independent of $w\in C(\kappa)$ as $X^\star(T)^+_{\mathbb{Q}}-\rho$ is a fundamental domain for the dot action of W on $X^\star(T)_{\mathbb{Q}}$.)

Proposition 3.7. *The cohomology complex $\mathrm{R}\Gamma(FL,\mathcal{L}_\kappa)^{bw(\nu)}$ is a perfect complex of $\mathcal{O}_F$-modules of amplitude $[\min_{w\in C(\kappa)}(d-\ell(w)),\max_{w\in C(\kappa)}(d-\ell(w))]$.*

Proof. Fix $w_1 \in C(\kappa)$ so that $\nu = (w_1^{-1}w_0)\kappa$. For $w \in W$, the weights occurring in $\mathrm{H}_w^{d-\ell(w)}(FL, \mathcal{L}_\kappa)$ are $\leq (w^{-1}w_0) \cdot \kappa = (w^{-1}w_1) \cdot \nu$. Moreover $w \in C(\kappa)$ if and only if $(w^{-1}w_1) \cdot \nu = \nu$. Hence if $w \notin C(\kappa)$, $\mathrm{H}_w^{d-\ell(w)}(FL, \mathcal{L}_\kappa)^{bw(\nu)} = 0$. Thus only the terms of the Cousin complex corresponding to $w \in C(\kappa)$ remain. □

Remark 3.8. In particular when $\kappa + \rho$ is regular so that $C(\kappa) = \{w\}$ consists of a single element, we have that $\mathrm{H}^\star(FL, \mathcal{L}_\kappa)^{bw(\nu)} = \mathrm{H}_w^\star(FL, \mathcal{L}_\kappa)^{bw(\nu)}$. We view this statement as a sort of analog of Coleman's classicality theorem, where the algebraic local cohomology groups $\mathrm{H}_w^\star$ play the role of the overconvergent cohomology groups introduced in this paper, and the big weight condition plays the role of the small slope condition.

We emphasize that the vanishing of Proposition 3.7 is characteristic independent. Of course in characteristic zero, the classical Borel-Weil-Bott theorem gives a precise description of $\mathrm{H}^\star(FL, \mathcal{L}_\kappa)$. For the sake of completeness, we explain how the Borel-Weil-Bott theorem may be deduced from the computation of $\mathrm{H}^\star(FL, \mathcal{L}_\kappa)$ via the Cousin complex and basic properties of category $\mathcal{O}$.

Theorem 3.9 ([Jan03], 5.5, corollary)**.** *Let $\kappa \in X^\star(T)$ then:*

(1) *If there exists no $w \in W$ such that $w \cdot \kappa$ is dominant then* $\mathrm{H}^i(FL, \mathcal{L}_\kappa) \otimes_{\mathcal{O}_F} F = 0$ *for all i,*
(2) *If there exists $w \in W$ such that $w \cdot \kappa$ is dominant, then there is a unique such w, and* $\mathrm{H}^i(FL, \mathcal{L}_\kappa) \otimes F = 0$ *if $\ell(w) \neq i$, while* $\mathrm{H}^{\ell(w)}(FL, \mathcal{L}_\kappa) \otimes F$ *is a highest weight $w \cdot \kappa$ representation.*

Proof. There is a famous sub-category of the category of $\mathcal{U}(\mathfrak{g})$-modules, called the category $\mathcal{O}$. See [Hum08], chapter 1 for the definition and properties of the category $\mathcal{O}$. We recall a number of basic results concerning the category $\mathcal{O}$ that will be used in the argument. The category $\mathcal{O}$ is abelian, Artinian and Noetherian. The simple objects are indexed by weights $\lambda \in X^\star(T) \otimes F$ and denoted by L_λ. If the module L_λ is finite dimensional then $\lambda \in X^\star(T) \otimes F$ is dominant. Moreover if $\lambda \in X^\star(T)^+$ then L_λ arises from the highest weight λ representation of G. For all $\lambda \in X^\star(T) \otimes F$ we also denote by $M_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} F(\lambda)$ the Verma module of weight λ. The simple module L_λ is the unique simple quotient of M_λ. The Grothendieck group of $\mathcal{O}$, denoted by $K(\mathcal{O})$, is the free module on the $[L_\lambda]$. We denote by $M \mapsto [M]$ the semi-simplification map from $\mathcal{O}$ to $K(\mathcal{O})$. In $K(\mathcal{O})$ we have $[M_\lambda] = \oplus_{w \cdot \lambda \leq \lambda} a(w \cdot \lambda, \lambda)[L_{w \cdot \lambda}]$ with $a(\lambda, \lambda) = 1$. Since any element $M \in \mathcal{O}$ has diagonalisable $\mathfrak{t}$-action, we can associate to M its formal character $\mathrm{ch}M$ which is an element of the group $\mathcal{X}$ of functions $X^\star(T) \otimes F \to \mathbb{Z}$. The character is additive on short exact sequence and we get a map $K(\mathcal{O}) \to \mathcal{X}$, $[M] \mapsto \mathrm{ch}[M]$. Moreover, this last map is a group injection. We denote by $\mathcal{X}_{\mathcal{O}}$ its image. Finally, any $\mathcal{U}(\mathfrak{g})$-module with diagonalisable $\mathfrak{t}$-action, and whose formal character belongs to $\mathcal{X}_{\mathcal{O}}$ is an object of $\mathcal{O}$.

It follows from lemma 3.5 that $\mathrm{ch}(\mathrm{H}^p_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) \otimes_{\mathcal{O}_F} F) = \oplus_{w, \ell(w)=p} \mathrm{ch}(M_{w \cdot \kappa})$. The Grothendieck-Cousin complex

$$0 \to \mathrm{H}^0_{Z_0/Z_1}(FL, \mathcal{L}_\kappa) \otimes F \to \cdots \to \mathrm{H}^d_{Z_d/Z_{d+1}}(FL, \mathcal{L}_\kappa) \otimes F \to 0$$

carries an action $\mathcal{U}(\mathfrak{g})$ by [Kem78], lemma 12.8, and is therefore a complex in the category $\mathcal{O}$. At this stage, we see that if none of the elements in the set $\{w \cdot \kappa\}$ is dominant, then none of the $\mathrm{H}^p_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) \otimes_{\mathcal{O}_F} F$ contains a finite dimensional subquotient. Otherwise there is a unique w such that $w\cdot\kappa$ is dominant. We see that for $p=\ell(w)$, $[\mathrm{H}^p_{Z_p/Z_{p+1}}(FL,\mathcal{L}_\kappa)\otimes_{\mathcal{O}_F}F]$ has a unique finite dimensional constituant (with multiplicity one) equal to $[L_{w\cdot\kappa}]$. On the other hand $[\mathrm{H}^i_{Z_i/Z_{i+1}}(FL,\mathcal{L}_\kappa)\otimes_{\mathcal{O}_F} F]$ has no finite dimensional constituant for $i\neq\ell(w)$. The cohomology groups $\mathrm{H}^i(FL,\mathcal{L}_\kappa)\otimes_{\mathcal{O}_F}F$ are finite dimensional because FL is proper. If none of the elements in the set $\{w\cdot\kappa\}$ is dominant, the cohomology is therefore trivial. If there is a unique w such that $w\cdot\kappa$ is dominant, we see that $\mathrm{H}^i(FL,\mathcal{L}_\kappa)\otimes_{\mathcal{O}_F}F=0$ if $i\neq\ell(w)$, and $\mathrm{H}^{\ell(w)}(FL,\mathcal{L}_\kappa)\otimes_{\mathcal{O}_F}F=L_{w\cdot\kappa}$. □

3.3. **Analytic geometry.** If $S\to\mathrm{Spec}\ \mathcal{O}_F$ is a finite type morphism of schemes, we let $\mathcal{S}=S\times_{\mathrm{Spec}\ \mathcal{O}_F}\mathrm{Spa}(F,\mathcal{O}_F)$ be the associated analytic adic space and $S_k=S\times_{\mathrm{Spec}\ \mathcal{O}_F}\mathrm{Spec}\ k$ be the special fiber. One can also consider $\mathcal{S}^{an}$, the analytification of the scheme $S\times_{\mathrm{Spec}\ \mathcal{O}_F}\mathrm{Spec}\ F$, and there is a map $\mathcal{S}\to\mathcal{S}^{an}$ which is an isomorphism when S is proper (see [Hub94], section 4). There is a continuous specialization map $\mathrm{sp}_{\mathcal{S}}:\mathcal{S}\to S_k$ and the preimage of a subset $U\subset S_k$ is denoted by $\mathrm{sp}_{\mathcal{S}}^{-1}(U)$. If U is a locally closed subset of S_k, we let $]U[_{\mathcal{S}}$ be the interior of $\mathrm{sp}^{-1}(U)$. This is an adic space, called the tube of U (see [Ber91]). The difference between $\mathrm{sp}_{\mathcal{S}}^{-1}(U)$ and $]U[_{\mathcal{S}}$ consists only of certain higher rank points. The tube $]U[_{\mathcal{S}}$ is the adic space associated to a "classical" rigid space, while $\mathrm{sp}_{\mathcal{S}}^{-1}(U)$ is not in general.

3.3.1. *The Iwahori decomposition.* Let $\mathcal{G}$ be the quasi-compact adic space associated to G and let $\mathrm{Iw}=]B_k[_{\mathcal{G}}$ be the Iwahori subgroup of $\mathcal{G}$.

For any root $\alpha\in\Phi$, we have an algebraic root space $U_\alpha\to\mathrm{Spec}\ \mathcal{O}_F$. We let $\mathcal{U}_\alpha$ be the corresponding quasi-compact adic space (isomorphic to a unit ball) and we let $\mathcal{U}^o_\alpha=]\{1\}[_{\mathcal{U}_\alpha}$ be the tube of the identity element (isomorphic to an increasing union of balls of radii $r<1$). We also let $\mathcal{U}^{an}_\alpha$ be the analytification of U_α (isomorphic to the affine line).

The following result gives a strong form of the Iwahori decomposition.

Proposition 3.10. *Let $\alpha_1,\cdots,\alpha_n$ be an enumeration of the roots in Φ. The product map*

$$\mathcal{T}\times\prod\mathcal{U}^{\star_i}_{\alpha_i}\to\mathrm{Iw}$$

is an isomorphism of analytic adic spaces, where $\mathcal{U}^{\star_i}_{\alpha_i}=\mathcal{U}_{\alpha_i}$ if $\alpha_i\in\Phi^+$ and $\mathcal{U}^{\star_i}_{\alpha_i}=\mathcal{U}^o_{\alpha_i}$ if $\alpha_i\in\Phi^-$.

Remark 3.11. The existence of a product decomposition $\mathcal{T}(K)\times\prod\mathcal{U}^{\star_i}_{\alpha_i}(K)\to\mathrm{Iw}(K)$ for K a discretely valued field is a consequence of Bruhat-Tits theory [Tit79], sect. 3.1.1.

Proof. We let $\mathfrak{Iw}$ be the formal group scheme obtained by completing the group G along the closed subscheme B_k. For each root α, we let $\mathfrak{U}_\alpha=\mathrm{Spf}\ \mathcal{O}_F\langle T\rangle$ be the formal one parameter subgroup and $\mathfrak{U}^o_\alpha=\mathrm{Spf}\ \mathcal{O}_F[[T]]$ be the formal completion of $\mathfrak{U}_\alpha$ at the point $T=0$. We also let $\mathfrak{T}$ be the formal completion of T along T_k. We consider the map of formal schemes: $\mathfrak{T}\times\prod\mathfrak{U}^{\star_i}_{\alpha_i}\to\mathfrak{Iw}$. We claim that this map is formally étale. Since it is an isomorphism on the associated reduced schemes because $T_k\times\prod_{\alpha\in\Phi^+}U_{\alpha,k}\to B_k$ is an isomorphism, we deduce that the map is an isomorphism. The associated map on the generic fiber (which is the map of the lemma) is an isomorphism. We are left to prove that the map is formally étale. Let

k' be a finite field extension of k. Let $(t, u_{\alpha_i}) \in \mathfrak{T} \times \prod \mathfrak{U}^{\star_i}_{\alpha_i}(k')$. Note that $u_{\alpha_i} = 1$ if $\alpha_i \in \Phi^-$. The map on Zariski tangent spaces is given by:

$$(t(1+\epsilon T), u_{\alpha_i}(1+\epsilon U_{\alpha_i})_{1\leq i\leq n}) \mapsto t(1+\epsilon T)\prod_i u_{\alpha_i}(1+\epsilon U_{\alpha_i})$$

where $(T, (U_{\alpha_i})_{1\leq i \leq n}) \in \mathrm{Lie}(T_k)\otimes_k k' \oplus_i \mathrm{Lie}(U_{\alpha_i})\otimes_k k'$, and there is an equality:

$$t(1+\epsilon T)\prod_i u_{\alpha_i}(1+\epsilon U_{\alpha_i}) = t\prod_i u_{\alpha_i}(1+\epsilon \mathrm{Ad}((\prod_i u_{\alpha_i})^{-1})(T) + \sum_{i=2}^{n+1}\mathrm{Ad}((\prod_{k=i}^{n} u_{\alpha_k})^{-1})U_{\alpha_{i-1}}).$$

Therefore, if we identify the tangent space of $\mathfrak{T}\times\prod \mathfrak{U}^{\star_i}_{\alpha_i}$ at (t, u_{α_i}) and the tangent space of $\mathfrak{Iw}$ at $t\prod u_{\alpha_i}$ with

$$\mathrm{Lie}(G_k)\otimes k' = \mathrm{Lie}(T_k)\otimes_k k' \oplus_i \mathrm{Lie}(U_{\alpha_i})\otimes_k k',$$

the map on tangent spaces is given by the endomorphism:

$$(T, (U_{\alpha_i})_{1\leq i\leq n}) \mapsto \mathrm{Ad}((\prod_i u_{\alpha_k})^{-1})(T) + \sum_{i=2}^{n+1}\mathrm{Ad}((\prod_{k=i}^{n} u_{\alpha_k})^{-1})(U_{\alpha_{i-1}}).$$

Observe that $(\prod_{k=i}^n u_{\alpha_k})^{-1} \in U(k')$ for all $1 \leq i \leq n$. By Chevalley's commutativity relations ([Ste16], chapter 6), we know that for any $u \in U(k')$, any $\alpha \in \Phi\cup\{0\}$ and any $v \in \mathrm{Lie}(G_k)_\alpha \otimes k'$, $Ad(u).v = v + w$ where $w \in \bigoplus_{\alpha'>\alpha}\mathrm{Lie}(G_k)_{\alpha'}\otimes k'$. A simple inductive argument proves that the map on tangent spaces is an isomorphism. □

For any $w \in {}^MW$, we now consider the tube of the Bruhat cell $]C_{w,k}[_{\mathcal{FL}} = \mathcal{P}\backslash\mathcal{P}w\mathrm{Iw}$, as well as the tube of the Schubert variety $]X_{w,k}[_{\mathcal{FL}}$ and the tube of $]Y_{w,k}[_{\mathcal{FL}}$. It follows from lemma 3.2 that $\mathcal{X}^w \hookrightarrow]Y_{w,k}[_{\mathcal{FL}}$.

3.3.2. *The tube of the Buhat cells.* We can now give a very precise description of the tube of the Bruhat cells.

Corollary 3.12. *For any $w \in {}^MW$, and for any ordering of the roots in Φ, we have an isomorphism of analytic spaces:*

$$\begin{array}{rcl}\prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^+}\mathcal{U}_\alpha \times \prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^-}\mathcal{U}^o_\alpha & \to &]C_{w,k}[_{\mathcal{FL}} \\ (u_\alpha)_{\alpha\in w^{-1}\Phi^{-,M}} & \mapsto & w\prod_\alpha u_\alpha\end{array}$$

where the product $\prod_\alpha u_\alpha$ is taken according to our fixed ordering.

Proof. This follows easily from proposition 3.15. □

We can also consider the analytification of U_w, $\mathcal{U}^{an}_w$. This open subset of $\mathcal{FL}$ contains $]C_{w,k}[_{\mathcal{FL}}$ and we have an isomorphism of analytic spaces:

$$\begin{array}{rcl}\prod_{\alpha\in(w^{-1}\Phi^{-,M})}\mathcal{U}^{an}_\alpha & \to & \mathcal{U}^{an}_w \\ (u_\alpha)_{\alpha\in w^{-1}\Phi^{-,M}} & \mapsto & w\prod_\alpha u_\alpha\end{array}$$

We now introduce certain quasi-Stein subspaces of $\mathcal{U}^{an}_w$ as well as some partial closures of them. If T is a topological space and $S \subseteq T$ is a subset, we let as usual $\overline{S}$ be the closure of S in T. We will use repeatedly the property that in a spectral space, the closure of a pro-constructible set is the set of all its specializations (see [Sch17], lemma 2.4).

We identify each $\mathcal{U}_\alpha$ with the unit ball of center 0 and coordinate u_α with its additive group law (the coordinate u_α is well defined up to multiplication by a unit). For all $m \in \mathbb{Q} \cup \{-\infty\}$ and all $\alpha \in \Phi$, we let $\mathcal{U}_{\alpha,m} = \{|.| \in \mathcal{U}_\alpha^{an},\ |u_\alpha| \leq |p^m|\}$ and $\mathcal{U}^o_{\alpha,m} = \cup_{m'>m}\mathcal{U}_{\alpha,m'}$. For all $m, n \in \mathbb{Q} \cup \{-\infty\}$, we let $]C_{w,k}[_{m,n}$ be the image of

$$w \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \mathcal{U}^o_{\alpha,m} \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \mathcal{U}_{\alpha,n} \to \mathcal{U}_w^{an},$$

For all $m, n \in \mathbb{Q}$, we let $]C_{w,k}[_{\overline{m},n}$ be the image of

$$w \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \overline{\mathcal{U}^o_{\alpha,m}} \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \mathcal{U}_{\alpha,n} \to \mathcal{U}_w^{an},$$

we let $]C_{w,k}[_{m,\overline{n}}$ be the image of

$$w \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \mathcal{U}^o_{\alpha,m} \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \overline{\mathcal{U}_{\alpha,n}} \to \mathcal{U}_w^{an},$$

and we let $]C_{w,k}[_{\overline{m},\overline{n}}$ be the image of

$$w \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \overline{\mathcal{U}^o_{\alpha,m}} \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \overline{\mathcal{U}_{\alpha,n}} \to \mathcal{U}_w^{an}.$$

Clearly, $]C_{w,k}[=]C_{w,k}[_{0,0}$, $]C_{w,k}[_{m,n} \subseteq]C_{w,k}[$ if and only if $m, n \geq 0$, and moreover then we have $]C_{w,k}[_{m,n} =]C_{w,k}[_{m,0} \cap]C_{w,k}[_{0,n}$, $]C_{w,k}[_{\overline{m},n} =]C_{w,k}[_{\overline{m},0} \cap]C_{w,k}[_{0,n}$ and $]C_{w,k}[_{m,\overline{n}} =]C_{w,k}[_{m,0} \cap]C_{w,k}[_{0,\overline{n}}$.

Remark 3.13. In the above formulas, we make the product for any ordering of the roots in $(w^{-1}\Phi^{-,M}) \cap \Phi^+$ or $(w^{-1}\Phi^{-,M}) \cap \Phi^-$. See [Ste16], lemma 17 for a justification that the order doesn't matter. We also point out that in our applications to Shimura varieties the unipotent radical of P is abelian so that the root groups U_α for $\alpha \in w^{-1}\Phi^{-,M}$ commute with each other.

Remark 3.14. For any $\alpha \in \Phi$, the closure $\overline{\mathcal{U}_\alpha}$ is the set of all specializations of points of $\mathcal{U}_\alpha$ in $\mathcal{U}_\alpha^{an}$ (or $\mathcal{G}^{an}$). Concretely, $\overline{\mathcal{U}_\alpha} \setminus \mathcal{U}_\alpha$ consists of a rank two point whose maximal generalization is the Gauss point of $\mathcal{U}_\alpha$ and which points towards ∞ (if we identify $\mathcal{U}_\alpha$ with a ball of center 0 and radius 1). Similarly $\overline{\mathcal{U}^o_\alpha}$ is $\mathrm{sp}^{-1}_{\mathcal{U}_\alpha}(\{1\})$. Concretely, $\overline{\mathcal{U}^o_\alpha} \setminus \mathcal{U}^o_\alpha$ consists of a rank two point whose maximal generalization is the Gauss point of $\mathcal{U}_\alpha$ and which points towards 0.

3.3.3. *Orbits of Cells.* We can give a more group theoretic description of certain of the above sets. We introduce some subgroups of $\mathcal{G}$.

For $m \in \mathbb{Q}_{\geq 0}$ we let $\mathcal{G}_{B,m}$ (resp. $\mathcal{G}_{U,m}$, $\mathcal{G}_{\overline{B},m}$, $\mathcal{G}_{\overline{U},m}$) for the affinoid subgroup of $\mathcal{G}$ of elements reducing to B (resp. U, $\overline{B}$, $\overline{U}$) mod p^m. We also define $\mathcal{G}^\circ_{B,m} = \cup_{m'>m}\mathcal{G}_{B,m'}$, $\mathcal{G}^\circ_{U,m} = \cup_{m'>m}\mathcal{G}_{B,m'}$.

Then for all $m, n \in \mathbb{Q}_{\geq 0}$ we let $\mathcal{G}_{m,n} = \mathcal{G}^\circ_{B,m} \cap \mathcal{G}_{\overline{B},n}$. In particular we note that $\mathcal{G}_{0,0} = \mathrm{Iw}$. We also let $\mathcal{G}^1_{m,n} = \mathcal{G}^\circ_{U,m} \cap \mathcal{G}_{\overline{U},n}$.

We would also like to introduce some "partial closures" of these groups. For all $m, n \in \mathbb{Q}_{\geq 0}$ we let $\mathcal{G}_{\overline{m},n} = \overline{\mathcal{G}^\circ_{B,m}} \cap \mathcal{G}_{\overline{B},n}$ and $\mathcal{G}^1_{\overline{m},n} = \overline{\mathcal{G}^\circ_{U,m}} \cap \mathcal{G}_{\overline{U},n}$, where the closures are taken inside $\mathcal{G}$. We note that $\mathcal{G}_{\overline{0},0} = \mathrm{sp}^{-1}(B_k)$, the closure of Iw in $\mathcal{G}$. Finally for all $m, n \in \mathbb{Q}_{\geq 0}$ with $n > 0$ we let $\mathcal{G}_{m,\overline{n}} = \mathcal{G}^\circ_{B,m} \cap \overline{\mathcal{G}_{\overline{B},n}}$ and $\mathcal{G}^1_{m,\overline{n}} = \mathcal{G}^\circ_{U,m} \cap \overline{\mathcal{G}_{\overline{U},n}}$ where again the closures are taken inside $\mathcal{G}$.

Note that the groups $\mathcal{G}_{m,n}$, $\mathcal{G}_{\overline{m},n}$, and $\mathcal{G}_{m,\overline{n}}$ all have the same rank 1 points. Moreover they all have Iwahori decompositions:

Proposition 3.15. *Let $w \in W$. Then for $m, n \in \mathbb{Q}_{\geq 0}$ and for $\mathcal{G}'$ one of $\mathcal{G}^\star_{m,n}$, $\mathcal{G}^\star_{\overline{m},n}$ or if $n > 0$, $\mathcal{G}^\star_{m,\overline{n}}$ with $\star \in \{\emptyset, 1\}$ the product map*

$$\mathcal{T} \cap \mathcal{G}' \times \prod_{\alpha_i \in w^{-1}\Phi^+} \mathcal{U}^{\star_i}_{\alpha_i} \prod_{\alpha_i \in w^{-1}\Phi^-} \mathcal{U}^{\star_i}_{\alpha_i} \to \mathcal{G}'$$

is an isomorphism, where if $\alpha_i \in \Phi^+$ then $\mathcal{U}^{\star_i}_{\alpha_i}$ is $\mathcal{U}_{\alpha_i,n}$ (resp. $\overline{\mathcal{U}_{\alpha_i,n}}$) if $\mathcal{G}' = \mathcal{G}^\star_{m,n}, \mathcal{G}^\star_{\overline{m},n}$ (resp. $\mathcal{G}' = \mathcal{G}^\star_{m,\overline{n}}$) while if $\alpha_i \in \Phi^-$ then $\mathcal{U}^{\star_i}_{\alpha_i}$ is $\mathcal{U}^o_{\alpha_i,m}$ (resp. $\overline{\mathcal{U}^o_{\alpha_i,m}}$) if $\mathcal{G}' = \mathcal{G}^\star_{m,n}, \mathcal{G}^\star_{m,\overline{n}}$. (resp. $\mathcal{G}' = \mathcal{G}^\star_{\overline{m},n}$.)

Proof. We give the argument for $\mathcal{G}_{\overline{0},0}$ and leave the general case to the reader. Since $\mathcal{G}_{\overline{0},0} = \mathrm{sp}^{-1}(B_k)$, and $B_k \hookrightarrow w^{-1}\overline{B_k}B_k w$, it follows that $\mathcal{G}_{\overline{0},0} \hookrightarrow \mathcal{T} \times \prod_{\alpha \in w^{-1}\Phi^-} \mathcal{U}_\alpha \prod_{\alpha \in w^{-1}\Phi^+} \mathcal{U}_\alpha$. We now let $g \in \mathcal{G}_{\overline{0},0}(K, K^+)$ for a field K and valuation ring $K^+ \subseteq K$. Let $\mathfrak{m}_{K^+}$ be the maximal ideal of K^+. By definition, $g \in G(K^+)$ and its image $\overline{g}$ in $G(K^+/\mathfrak{m}^+_K)$ lies in B. Let $g = t\prod_{\alpha \in w^{-1}\Phi^+} u_\alpha \prod_{\alpha \in w^{-1}} u_\alpha$. We have $\overline{g} = \overline{t}\prod_{\alpha \in w^{-1}\Phi^+} \overline{u_\alpha} \prod_{\alpha \in w^{-1}} \overline{u_\alpha}$ and we find that $\overline{u_\alpha} = 1$ if $\alpha \in \Phi^-$ by unicity of the decomposition. □

Remark 3.16. For the group $\mathcal{G}_{0,0} = \mathrm{Iw}$, the decomposition holds for any ordering of the root $\alpha \in \Phi$ by proposition 3.15. We don't know if this property holds for $\mathcal{G}_{\overline{0},0}$ for example.

As a consequence we have that for all $m, n \in \mathbb{Q}_{\geq 0}$, $]C_{w,k}[_{m,n} = \mathcal{P}\backslash\mathcal{P}w\mathcal{G}_{m,n} = \mathcal{P}\backslash\mathcal{P}w\mathcal{G}^1_{m,n}$, $]C_{w,k}[_{\overline{m},n} = \mathcal{P}\backslash\mathcal{P}w\mathcal{G}_{\overline{m},n} = \mathcal{P}\backslash\mathcal{P}w\mathcal{G}^1_{\overline{m},n}$ and if $n > 0$, $]C_{w,k}[_{m,\overline{n}} = \mathcal{P}\backslash\mathcal{P}w\mathcal{G}_{m,\overline{n}} = \mathcal{P}\backslash\mathcal{P}w\mathcal{G}^1_{m,\overline{n}}$.

Lemma 3.17. *For all $m \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{>0}$, the groups $\mathcal{G}^1_{m+k,k}$, $\mathcal{G}^1_{\overline{m+k},k}$ and $\mathcal{G}^1_{m+k,\overline{k}}$ are normalized by $\mathcal{G}_{m,0}$.*

Proof. We can find a closed embedding $G \hookrightarrow \mathrm{GL}_r$ and a Borel $B_{\mathrm{GL_r}}$ of GL_r with the property that $B = G \cap B_{GL_n}$. Indeed, first consider a faithful representation of G into GL_r, then consider the action of B on the flag variety of Borels of GL_r. This action of a solvable group on a proper scheme must have a fixed point B_{GL_r}, and then $B \subseteq G \cap B_{GL_r}$. But then we must have $B = G \cap B_{GL_r}$ as the later is a solvable subgroup of G containing B. Therefore, the problem is reduced to the case of the group GL_r. We now consider certain sub-algebras of the algebra of $r \times r$ matrices $\mathcal{M}^{an}_r$. For all $s \in \mathbb{Q}$, we let $\mathbb{B}(0, s)$ the (quasi-compact) ball of center 0 and radius s, and $\mathbb{B}^o(0, s) = \cup_{s' < s}\mathbb{B}(0, s')$ be the "open" ball of radius s (which is a Stein space). For any $m \geq 0$, we let:

$$\mathrm{Lie}_{m,0}(S, S^+) =$$

$$\{(a_{i,j}) \in \mathcal{M}^{an}_r(S, S^+),\ a_{i,j} \in \mathbb{B}^o(0, |p^m|)(S, S^+) \text{ if } i \geq j, a_{i,j} \in \mathbb{B}(0, 1)(S, S^+) \text{ if } i \leq j\},$$

$$\mathrm{Lie}_{\overline{m},0}(S, S^+) =$$

$$\{(a_{i,j}) \in \mathcal{M}^{an}_r(S, S^+), a_{i,j} \in \overline{\mathbb{B}^o(0, |p^m|)}(S, S^+) \text{ if } i \geq j, a_{i,j} \in \mathbb{B}(0, 1)(S, S^+) \text{ if } i \leq j\},$$

$$\mathrm{Lie}_{m,\overline{0}}(S, S^+) =$$

$$\{(a_{i,j}) \in \mathcal{M}^{an}_r(S, S^+),\ a_{i,j} \in \mathbb{B}^o(0, |p^m|)(S, S^+) \text{ if } i \geq j\ a_{i,j} \in \overline{\mathbb{B}(0, 1)}(S, S^+) \text{ if } i \leq j\}.$$

We claim that these algebras are stable under the adjoint action of $\mathcal{G}_{m,0}$. Since $\mathrm{Lie}_{m,0}$ is the Lie algebra of $\mathcal{G}^1_{m,0}$, and $\mathcal{G}_{m,0}$ normalizes $\mathcal{G}^1_{m,0}$, this case follows easily. The other cases are elementary to check by hand. A typical element of $\mathcal{G}^1_{m+k,k}$ (resp. $\mathcal{G}^1_{\overline{m+k},k}$, resp. $\mathcal{G}^1_{m+k,\overline{k}}$) writes $(1+p^k g)$ where $g \in \mathrm{Lie}_{m,0}$ (resp. $\mathrm{Lie}_{\overline{m},0}$, resp. $\mathrm{Lie}_{m,\overline{0}}$). For $h \in \mathcal{G}_{m,0}$, we have $h(1+p^kg)h^{-1} = 1 + p^k \mathrm{ad}(h).g$. □

Lemma 3.18. *Let $w \in {}^MW$. Let $m \geq 0$, and let $K_p \subset \mathcal{G}_{m,0}$ be a profinite subgroup. For all $k \geq 1$, the sets*

$$]C_{w,k}[_{m+k,k}K_p,\ ]C_{w,k}[_{\overline{m+k},k}K_p \text{ and }]C_{w,k}[_{m+k,\overline{k}}K_p$$

are a finite disjoint union of translates of the form

$$]C_{w,k}[_{m+k,k}h,\ ]C_{w,k}[_{\overline{m+k},k}h, \text{ and respectiveley }]C_{w,k}[_{m+k,\overline{k}}h \text{ for } h \in K_p.$$

Proof. We prove the first statement, the others are identical. For $h \in K_p$ we have $]C_{w,k}[_{m+k,k}h = \mathcal{P}\backslash\mathcal{P}w\mathcal{G}^1_{m+k,k}h = \mathcal{P}\backslash\mathcal{P}wh\mathcal{G}^1_{m+k,k}$ by Lemma 3.17. This proves that if for $h, h' \in K_p$, $]C_{w,k}[_{m+k,k}h \cap]C_{w,k}[_{m+k,k}h' \neq \emptyset$, then $]C_{w,k}[_{m+k,k}h =]C_{w,k}[_{m+k,k}h'$. Therefore, $]C_{w,k}[_{m+k,k}K_p$ is a disjoint union. This disjoint union is finite because $\mathcal{G}_{m+k,k} \cap K_p$ is of finite index in K_p. □

We now consider certain intersections.

Lemma 3.19. *Let $w \in {}^MW$. Let $m, n \in \mathbb{Z}_{\geq 0}$ and let K_p be a subgroup of $\mathcal{G}_{m,0}$. Then we have:*

$$\begin{aligned}]C_{w,k}[_{m,0}K_p \cap]C_{w,k}[_{0,n}K_p &=]C_{w,k}[_{m,n}K_p, \\ ]C_{w,k}[_{m,\overline{0}}K_p \cap]C_{w,k}[_{0,\overline{n}}K_p &=]C_{w,k}[_{m,\overline{n}}K_p, \\ ]C_{w,k}[_{\overline{m},0}K_p \cap]C_{w,k}[_{\overline{0},n}K_p &=]C_{w,k}[_{\overline{m},n}K_p. \end{aligned}$$

Proof. We only check the first statement as the others are identical. As $K_p \subset \mathcal{G}_{m,0}$, $]C_{w,k}[_{m,0}K_p =]C_{w,k}[_{m,0}$. Let $x \in]C_{w,k}[_{m,0}K_p \cap]C_{w,k}[_{0,n}K_p$. Then, there exists $h \in K_p$, $xh \in]C_{w,k}[_{m,0} \cap]C_{w,k}[_{0,n} =]C_{w,k}[_{m,n}$. The reverse inclusion is trivial. □

3.3.4. *Intersections and unions of cells.* We prove a few more results concerning the intersections and unions of various subsets of $\mathcal{FL}$ we have introduced so far.

Lemma 3.20. *Let $w, w' \in {}^MW$. Assume that $\ell(w) \leq \ell(w')$ and $w \neq w'$. Then:*

(1) $\overline{]X_{w,k}[} \cap]Y_{w',k}[= \emptyset$,
(2) $]X_{w,k}[\cap \overline{]Y_{w',k}[} = \emptyset$.

Proof. We prove the first point. Note that $\overline{]X_{w,k}[} = \mathrm{sp}^{-1}(X_{w,k})$ and $]Y_{w',k}[= \mathrm{sp}^{-1}(Y_{w',k})$. Thus the first point follows from the fact that $X_{w,k} \cap Y_{w',k} = \emptyset$. We prove the second point. Since $]Y_{w',k}[$ is quasi-compact open, $\overline{]Y_{w',k}[}$ is the set of all its specializations. Therefore, both $\overline{]Y_{w',k}[}$ and $]X_{w,k}[$ are stable under generalization. But they have no common rank one point, again as $X_{w,k} \cap Y_{w',k} = \emptyset$. □

Lemma 3.21. (1) *We have $\overline{]C_{w,k}[} =]C_{w,k}[_{\overline{0},\overline{0}} \subseteq \mathcal{U}^{an}_w$,*

(2) *We have $]C_{w,k}[_{0,-\infty} \subseteq]X_{w,k}[$.*
(3) *We have $]C_{w,k}[_{-\infty,0} \subseteq]Y_{w,k}[$.*
(4) *We have $\overline{]Y_{w,k}[} \cap]X_{w,k}[=]C_{w,k}[_{0,\overline{0}} \subseteq \overline{]C_{w,k}[}$.*
(5) *We have $]Y_{w,k}[\cap \overline{]X_{w,k}[} =]C_{w,k}[_{\overline{0},0} \subseteq \overline{]C_{w,k}[}$.*

Proof. We begin with the first point. Let $Z = FL \setminus U_w$, a closed subset. Let Z_k be the special fiber of Z. As $C_{w,k} \subset U_{w,k}$, $]C_{w,k}[\cap]Z_k[=\emptyset$ and therefore $]C_{w,k}[\subseteq]Z_k[^c$ and so $\overline{]C_{w,k}[} \subseteq]Z_k[^c$ as $]Z_k[^c$ is closed. It remains to prove that $]Z_k[^c \subseteq \mathcal{U}_w^{an}$. To see this, we note that $\mathcal{FL} = \mathcal{U}_w^{an} \cup]Z_k[$ is a covering of $\mathcal{FL}$.

We now prove the second point. We observe that $w\prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\Phi^+}\mathcal{U}_\alpha^{an} = \mathcal{C}_w^{an} \subseteq \mathcal{X}_w \subseteq]X_{w,k}[$ and since $]X_{w,k}[$ is invariant under multiplication by the Iwahori subgroup,

$$w \prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^+} \mathcal{U}_\alpha^{an} \times \prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^-} \mathcal{U}_\alpha^{o} \subseteq]X_{w,k}[.$$

We check the third point. We observe that $w\prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\Phi^-}\mathcal{U}_\alpha^{an} = \mathcal{C}^{w,an} \subseteq \mathcal{X}^w \subseteq]Y_{w,k}[$ and again the conclusion follows because $]Y_{w,k}[$ is invariant under multiplication by the Iwahori subgroup.

We prove the fourth point. Since $]Y_{w,k}[$ is quasi-compact, it is constructible, and therefore $\overline{]Y_{w,k}[}$ is the set of all specializations of $]Y_{w,k}[$ in $\mathcal{FL}$. Let $x \in \overline{]Y_{w,k}[}\cap]X_{w,k}[$ and let y be the maximal generalization of x in $\mathcal{FL}$. Then $y \in]Y_{w,k}[\cap]X_{w,k}[=]C_{w,k}[$. The subset of $\overline{]C_{w,k}[}$ consisting of points whose maximal generalization is in $]C_{w,k}[$ is exactly $w\prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^+}\overline{\mathcal{U}_\alpha} \times \prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^-}\mathcal{U}_\alpha^o$. This proves that $\overline{]Y_{w,k}[}\cap]X_{w,k}[$ is included in that set. The converse inclusion follows easily from the second and third points. We prove the last point. Since $\overline{]X_{w,k}[} = \mathrm{sp}^{-1}(X_{w,k})$ and $]Y_{w,k}[= \mathrm{sp}^{-1}(Y_{w,k})$, we deduce that $]Y_{w,k}[\cap \overline{]X_{w,k}[} = \mathrm{sp}^{-1}(C_{w,k}) \subseteq \overline{]C_{w,k}[}$ has exactly the announced description. □

Lemma 3.22. (1) *We have* $]X_{w,k}[=]C_{w,k}[_{0,\overline{0}}\bigcup\cup_{w'\leq w}]X_{w',k}[$.
(2) *We have* $]Y_{w,k}[=]C_{w,k}[_{\overline{0},0}\bigcup\cup_{w'\geq w}]Y_{w',k}[$.

Proof. We prove the first point and the direct inclusion. We first observe that since $X_{w',k}$ is closed, $]X_{w',k}[$ is a finite union of Stein spaces. Therefore $X_{w',k}$ is stable under specialization. Let $x \in]X_{w,k}[$. If its maximal generalization is in $]X_{w',k}[$ for some $w' \leq w$, then actually $x \in]X_{w',k}[$. Otherwise, $x \in \overline{]C_{w,k}[}\cap]X_{w,k}[=]C_{w,k}[_{0,\overline{0}}$ by lemma 3.21. The converse inclusion inclusion follows from the same lemma. We prove the second point. We note that $]Y_{w,k}[= \mathrm{sp}^{-1}(Y_{w,k})$ and therefore,

$$]Y_{w,k}[= \mathrm{sp}^{-1}(C_{w,k})\cap]Y_{w,k}[\bigcup\cup_{w'\geq w}]Y_{w',k}[.$$

By lemma 3.21 $\mathrm{sp}^{-1}(C_{w,k})\cap]Y_{w,k}[\subset]C_{w,k}[_{\overline{0},0}$. The reverse inclusion follows also from the lemma. □

3.4. **Dynamics.** Let $v : F \to \mathbb{R}\cup\{+\infty\}$ be the p-adic valuation normalized by $v(p) = 1$. We consider certain sub-semigroups of $T(F)$. We let $T^+(F) = \{t \in T(F), v(\alpha(t)) \geq 0,\ \forall\alpha\in\Phi^+\}$, $T^{++}(F) = \{t \in T(F), v(\alpha(t)) > 0,\ \forall\alpha\in\Phi^+\}$, $T^-(F) = \{t \in T(F), v(\alpha(t)) \leq 0,\ \forall\alpha\in\Phi^+\}$, $T^{--}(F) = \{t \in T(F), v(\alpha(t)) < 0,\ \forall\alpha\in\Phi^+\}$.

Lemma 3.23. (1) *If* $t \in T^+(F)$, $]X_{w,k}[.t \subseteq]X_{w,k}[$ *for all* $w \in {}^MW$.
(2) *If* $t \in T^-(F)$, $]Y_{w,k}[.t \subseteq]Y_{w,k}[$ *for all* $w \in {}^MW$.

Proof. By [Hub93], corollary 4.2, we reduce to check the inclusions on rank 1 points. For the first point, it suffices therefore to prove that $]C_{w,k}[.t \subseteq]X_{w,k}[$. Let $wbu \in]C_{w,k}[$ with $b \in \mathcal{B}$ and $u \in \prod_{\alpha\in\Phi^-}\mathcal{U}_\alpha^o$. We find that $wbut = wt^{-1}btt^{-1}ut$. Now, $\mathrm{Ad}(t^{-1})u \in \prod_{\alpha\in\Phi^-}\mathcal{U}_\alpha^o$, while $\mathrm{Ad}(t^{-1})b \in \mathcal{B}^{an}$. In particular $w\mathrm{Ad}(t^{-1})b \in \mathcal{X}_w \subseteq$

$]X_{w,k}[_{\mathcal{FL}}$. Therefore $w\mathrm{Ad}(t^{-1})b\mathrm{Ad}(t^{-1})u \in]X_{w,k}[$. For the second point, it suffices to prove that $]C_{w,k}[.t \subseteq]Y_{w,k}[$. Let $wub \in]C_{w,k}[$ with $b \in \mathcal{B}$ and $u \in \overline{\mathcal{U}}^o$. We find that $wubt = wt^{-1}utt^{-1}bt$. Now, $\mathrm{Ad}(t^{-1})b \in \mathcal{B}$, while $\mathrm{Ad}(t^{-1})u \in \overline{\mathcal{B}}^{an}$. In particular, $w\mathrm{Ad}(t^{-1})u \in \mathcal{X}^w \subseteq]Y_{w,k}[_{\mathcal{FL}}$. Therefore $w\mathrm{Ad}(t^{-1})u\mathrm{Ad}(t^{-1})b \in]Y_{w,k}[$. □

For $t \in T^{++}(F)$, we define $\min(t) = \min_{\alpha\in\Phi^+} v(\alpha(t))$ and $\max(t) = \max_{\alpha\in\Phi^+} v(\alpha(t))$. For $t \in T^{--}(F)$, we let $\min(t) = \min_{\alpha\in\Phi^-} v(\alpha(t))$ and $\max(t) = \max_{\alpha\in\Phi^-} v(\alpha(t))$. We note that $\min(t) > 0$ and that $\min(t) = \min(t^{-1})$ and $\max(t) = \max(t^{-1})$ for $t \in T^{++}(F)$.

Lemma 3.24. (1) *Let $t \in T^{++}(F)$. For all $w \in {}^MW$, $m, n \in \mathbb{Q}$, we have*

$$\begin{aligned}]C_{w,k}[_{m+\max(t),n-\min(t)} &\subseteq]C_{w,k}[_{m,n}.t \subseteq]C_{w,k}[_{m+\min(t),n-\max(t)} \\ ]C_{w,k}[_{\overline{m+\max(t)},n-\min(t)} &\subseteq]C_{w,k}[_{\overline{m},n}.t \subseteq]C_{w,k}[_{\overline{m+\min(t)},n-\max(t)}, \\ ]C_{w,k}[_{m+\max(t),\overline{n-\min(t)}} &\subseteq]C_{w,k}[_{m,\overline{n}}.t \subseteq]C_{w,k}[_{m+\min(t),\overline{n-\max(t)}}. \end{aligned}$$

(2) *Let $t \in T^{--}(F)$. For all $w \in {}^MW$ and $m, n \in \mathbb{Q}$, we have*

$$\begin{aligned}]C_{w,k}[_{m-\min(t),n+\max(t)} &\subseteq]C_{w,k}[_{m,n}.t \subseteq]C_{w,k}[_{m-\max(t),n+\min(t)} \\ ]C_{w,k}[_{\overline{m-\min(t)},n+\max(t)} &\subseteq]C_{w,k}[_{\overline{m},n}.t \subseteq]C_{w,k}[_{\overline{m-\max(t)},n+\min(t)}, \\ ]C_{w,k}[_{m-\min(t),\overline{n+\max(t)}} &\subseteq]C_{w,k}[_{m,\overline{n}}.t \subseteq]C_{w,k}[_{m-\max(t),\overline{n+\min(t)}}. \end{aligned}$$

Proof. Easy and left to the reader. □

3.5. **Dynamics of correspondences.**

3.5.1. *Certain compact open subgroups.* We now assume that the group $G_F = G \times_{\mathrm{Spec}\ \mathcal{O}_F} \mathrm{Spec}\ F$ is defined over $\mathbb{Q}_p$ and quasi-split. Therefore, we have a reductive group $G_{\mathbb{Q}_p}$ with borel $B_{\mathbb{Q}_p}$. The group $G_{\mathbb{Q}_p}$ splits over the extension F of $\mathbb{Q}_p$. Moreover, we have a reductive model G of G_F over $\mathrm{Spec}\ \mathcal{O}_F$. The Borel $B_{\mathbb{Q}_p}$ base changes to a Borel B_F of G_F which extends to a Borel B of G. Simlarly, we have a maximal torus $T_{\mathbb{Q}_p} \subseteq B_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$, and its base change T_F extends to a maximal (split) torus of G. We will often drop the subscripts $\mathbb{Q}_p$ or F when the context is clear.

Remark 3.25. Starting from the next section, we will consider a Shimura datum (G, X), where G is a reductive group over $\mathbb{Q}$. The group $G_{\mathbb{Q}_p}$ that we consider here will be the base change to $\mathbb{Q}_p$ of the group G which is part of the Shimura datum. We appologize for this slightly inconsistant notation.

We let $T(\mathbb{Z}_p)$ be the maximal compact subgroup of $T(\mathbb{Q}_p)$. Note that $T(\mathbb{Z}_p) = T(\mathcal{O}_F) \cap T(\mathbb{Q}_p)$. We have an exact sequence $0 \to \mathcal{O}_F^\times \to F^\times \xrightarrow{v} \mathbb{Q}$ and tensoring with $X_\star(T)$ and taking Galois invariants, we obtain the sequence: $0 \to T(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \xrightarrow{v} X_\star(T^d) \otimes \mathbb{Q}$ where T^d stands for the maximal split torus inside T and $X_\star(T^d)$ for its co-character group. The image of $T(\mathbb{Q}_p)$ in $X_\star(T^d) \otimes \mathbb{Q}$ is easily seen to be a $\mathbb{Z}$-lattice.

We let $T^+ = T(\mathbb{Q}_p) \cap T^+(F)$, $T^{++} = T(\mathbb{Q}_p) \cap T^{++}(F)$, $T^- = T(\mathbb{Q}_p) \cap T^-(F)$, and $T^{--} = T(\mathbb{Q}_p) \cap T^{--}(F)$. These are monoids in $T(\mathbb{Q}_p)$ and one proves easily that they generate $T(\mathbb{Q}_p)$ (since there are regular elements in the maximal split torus T^d).

We will now consider certain compact open subgroups of $G(\mathcal{O}_F)$. For all $m \in \mathbb{Z}_{\geq 0}$, we let $\tilde{K}_{p,m} \subseteq G(\mathcal{O}_F)$ be the preimage of $B(\mathcal{O}_F/\varpi^m)$ under the map $G(\mathcal{O}_F) \to G(\mathcal{O}_F/\varpi^m)$. We observe that $\tilde{K}_{p,0} = G(\mathcal{O}_F)$ and $\tilde{K}_{p,1}$ is the Iwahori subgroup of $G(\mathcal{O}_F)$ with respect to the Borel $B(\mathcal{O}_F)$. For $b \in \mathbb{Z}_{\geq 0}$ we let $\tilde{K}_{p,b,b} \subseteq G(\mathcal{O}_F)$ be the preimage of $U(\mathcal{O}_F/\varpi^b)$ under the map $G(\mathcal{O}_F) \to G(\mathcal{O}_F/\varpi^b)$. Finally for $m \geq b \geq 0$ we let $\tilde{K}_{p,m,b} = \tilde{K}_{p,m} \cap \tilde{K}_{p,b,b}$. In other words $\tilde{K}_{p,m,b}$ is the subgroup of $G(\mathcal{O}_F)$ of elements whose reduction mod ϖ^m lies in B, and whose reduction mod ϖ^b lies in U. We note that we have $\tilde{K}_{p,m,b} \subset \mathcal{G}_{m-1,0}$.

For $m \geq b \geq 0$ and $m > 0$, the groups $\tilde{K}_{p,m,b}$ have an Iwahori decomposition, in the sense that the product map

$$\overline{\tilde{U}}_m \times \tilde{T}_b \times U(\mathcal{O}_F) \to K_{p,m,b}$$

is a bijection, where $\tilde{T}_b = \ker(T(\mathcal{O}_F) \to T(\mathcal{O}_F/\varpi^b\mathcal{O}_F))$ and $\overline{\tilde{U}}_m = \ker(\overline{U}(\mathcal{O}_F) \to \overline{U}(\mathcal{O}_F/\varpi^m))$.

We now let $K_{p,m,b} = G(\mathbb{Q}_p) \cap \tilde{K}_{p,m,b}$. This is a compact open subgroup of $G(\mathbb{Q}_p)$.

For $m \geq b \geq 0$ and $m > 0$, the groups $K_{p,m,b}$ have an Iwahori decomposition, in the sense that the product map

$$\overline{U}_m \times T_b \times U(\mathbb{Z}_p) \to K_{p,m,b}$$

is a bijection, where $T_b = \tilde{T}_b \cap T(\mathbb{Q}_p)$ and $\overline{U}_m = \overline{\tilde{U}}_m \cap \overline{U}(\mathbb{Q}_p)$, $U(\mathbb{Z}_p) = U(\mathcal{O}_F) \cap U(\mathbb{Q}_p)$.

Remark 3.26. Let L be a local field with ring of integers $\mathcal{O}_L$, maximal ideal $\mathfrak{m}_{\mathcal{O}_L}$ and finite residue field. Let H be an unramified reductive group over Spec L. Then H admits a quasi-split reductive model $\tilde{H} \to$ Spec $\mathcal{O}_L$. Let $\tilde{B} \hookrightarrow \tilde{H}$ be a Borel subgroup. The Iwahori subgroup of $G(L)$ attached to $\tilde{B}$ is by definition the subgroup of $\tilde{H}(\mathcal{O}_L)$ of elements with reduction in $\tilde{B}(\mathcal{O}_L/\mathfrak{m}_{\mathcal{O}_L})$. We also note that the Iwahori subgroups of $G(L)$ are all conjugated. If we don't assume that H is unramified (and therefore to have a reductive model over Spec $\mathcal{O}_L$), there is a notion of Iwahori subgroup defined using Bruhat-Tits buildings. See the appendix of [PR08]. This definition is rather involved. In the situation that $H = \mathrm{Res}_{L'/L}H'$ for a finite extension L' of L and H' is an unramified reductive group over Spec L', then an Iwahori of $H(L)$ is the same thing as an Iwahori of $H'(L')$. This follows from the natural identification between the Bruhat-Tits buildings of H and H' (see [Pra01], page 172 for example). Going back to our situation, it is not clear to us wether the group $K_{p,1,0}$ is always an Iwahori subgroup of $G(\mathbb{Q}_p)$. In the important case that $G = \mathrm{Res}_{K/\mathbb{Q}_p}G_1$ where G_1 is an unramified reductive group over Spec K and K is a finite extension of $\mathbb{Q}_p$, the group $K_{p,1,0}$ is indeed an Iwahori by the above discussion. In the general case, we found it convenient to work with $K_{p,1,0}$, and the problem of deciding wether $K_{p,1,0}$ is an Iwahori or not has no influence on the results of this paper.

3.5.2. *Change of group.* In this paragraph we make a short digression that will be useful when we deal with abelian type Shimura varieties. Assume that we have an epimorphism of reductive groups $G_{\mathbb{Q}_p} \to G'_{\mathbb{Q}_p}$ with central kernel. This implies that G and G' have the same adjoint group G^{ad}. We assume that these group split over F and we fix models over $\mathcal{O}_F$, that we denote G and G' together with a map $G \to G'$ over Spec $\mathcal{O}_F$, extending the map $G_F \to G'_F$. We also assume that $G_{\mathbb{Q}_p}$ and $G'_{\mathbb{Q}_p}$ are quasi split and pick Borel B and B' defined over $\mathbb{Q}_p$ such that

$B \to B'$. We can therefore define compact open subgroups $K_{p,m,b} \subseteq G(\mathbb{Q}_p)$ and $K'_{p,m,b} \subseteq G'(\mathbb{Q}_p)$ as in section 3.5.1.

Lemma 3.27. *The groups $K_{p,m,b}$ and $K'_{p,m,b}$ have the same image in $G^{ad}(\mathbb{Q}_p)$.*

Proof. This follows from the Iwahori decomposition of these groups. Since $G_{\mathbb{Q}_p} \to G'_{\mathbb{Q}_p}$ is an epimorphism with central kernel, the induced map on root groups are isomorphisms. □

3.5.3. *Hecke algebras.* We let $\mathcal{H}_{p,m,b}$ be the Hecke algebras $\mathbb{Z}[K_{p,m,b}\backslash G(\mathbb{Q}_p)/K_{p,m,b}]$. We denote by $\mathcal{H}^+_{p,m,b}$ the sub-algebra generated by the double cosets $[K_{p,m,b}tK_{p,m,b}]$ with $t \in T^+$ and by $\mathcal{H}^{++}_{p,m,b}$ the ideal generated by $[K_{p,m,b}tK_{p,m,b}]$ with $t \in T^{++}$. We define similarly $\mathcal{H}^-_{p,m,b}$ and $\mathcal{H}^{--}_{p,m,b}$.

Lemma 3.28. *For all $m \geq b \geq 0$ with $m > 0$, the map $t \mapsto [K_{p,m,b}tK_{p,m,b}]$ induces isomorphisms $\mathbb{Z}[T^+/T_b] \to \mathcal{H}^+_{p,m,b}$ and $\mathbb{Z}[T^-/T_b] \to \mathcal{H}^-_{p,m,b}$.*

Proof. This is [Cas], lem. 4.1.5. (Alternatively it can be deduced from lemma 4.16 below.) □

3.5.4. *Action of correspondences on the flag variety.* Let $K_p \subseteq G_0(\mathbb{Q}_p)$ be a compact open subgroup.

We consider the quotient space $\mathcal{FL}/K_p$. This space carries an action by correspondences of double cosets K_pgK_p for $g \in G_0(\mathbb{Q}_p)$. Namely, given $xK_p \in \mathcal{FL}/K_p$, we let $xK_p.K_pgK_p = \{x\tilde{u}gK_p,\ u \in K_p/(gK_pg^{-1} \cap K_p),\ \tilde{u} \in K_p \text{ lifts } u\}$.

We now consider a compact open $K_p = K_{p,m',b}$ for $m' \geq b \geq 0$ and $m' > 0$.

Lemma 3.29. *Let $w \in {}^MW$.*

1. *Let $t \in T^+$. The sequence $\{]X_{w,k}[.(K_ptK_p)^m\}_{m\geq 0}$ is nested.*
2. *Let $t \in T^-$. The sequence $\{]Y_{w,k}[.(K_ptK_p)^n\}_{n\geq 0}$ is nested.*
3. *Let $t \in T^{++}$ and $m \geq 0$. We have*

$$]C_{w,k}[_{m\max(t),\overline{-m\min(t)}}.K_p \subseteq$$

$$]X_{w,k}[.(K_ptK_p)^m \subseteq]C_{w,k}[_{m\min(t),\overline{-m\max(t)}}.K_p \bigcup \cup_{w'<w}]X_{w',k}[.$$

4. *Let $t \in T^{--}$ and $n \geq 0$. We have*

$$]C_{w,k}[_{\overline{-n\min(t)},n\max(t)}.K_p \subseteq$$

$$]Y_{w,k}[.(K_ptK_p)^n \subseteq]C_{w,k}[_{\overline{-n\max(t)},n\min(t)}.K_p \bigcup \cup_{w'>w}]Y_{w',k}[.$$

5. *Let $t \in T^{++}$. For all $m, n \in \mathbb{Z}_{\geq 0}$,*

$$]X_{w,k}[.(K_ptK_p)^m \cap \overline{]Y_{w,k}[}.(K_pt^{-1}K_p)^n \subseteq]C_{w,k}[_{m\min(t),\overline{0}}K_p \cap]C_{w,k}[_{0,\overline{n\min(t)}}K_p.$$

6. *Let $t \in T^{--}$. For all $m, n \in \mathbb{Z}_{\geq 0}$,*

$$\overline{]X_{w,k}[}.(K_pt^{-1}K_p)^m \cap]Y_{w,k}[.(K_ptK_p)^n \subseteq]C_{w,k}[_{\overline{m\min(t)},0}K_p \cap]C_{w,k}[_{\overline{0},n\min(t)}K_p.$$

Proof. We observe that $]X_{w,k}[.(K_ptK_p)^m =]X_{w,k}[t^mK_p$ and $]Y_{w,k}[.(K_ptK_p)^m =]Y_{w,k}[t^mK_p$, using the very definition of the action, and also noting that $(K_ptK_p)^m = K_pt^mK_p$ by lemma 3.28. Therefore, the first and second points follow from lemma 3.23.

We note that $]X_{w,k}[=]C_{w,k}[_{0,\overline{0}}\bigcup \cup_{w'<w}]X_{w',k}[$ and $]Y_{w,k}[=]C_{w,k}[_{\overline{0},0}\bigcup \cup_{w'>w}]Y_{w',k}[$ by lemma 3.22. The third and fourth point follow from this, the first two points, and lemma 3.24.

We know that $]X_{w,k}[\cap\overline{]Y_{w,k}[} =]C_{w,k}[_{0,\overline{0}}$ by lemma 3.21. Moreover, $]C_{w,k}[_{0,\overline{0}}\cap]X_{w',k}[=\emptyset$ if $w'\neq w$ and $]C_{w,k}[_{0,\overline{0}}\cap]Y_{w',k}[=\emptyset$ if $w'\neq w$ (because $]C_{w,k}[_{0,\overline{0}}$, $]X_{w',k}[$ and $]Y_{w,k}[$ are all stable under generalization and they have no common rank one point). Therefore,

$$]X_{w,k}[.(K_ptK_p)^m\cap\overline{]Y_{w,k}[}.(K_pt^{-1}K_p)^n\subseteq$$
$$]C_{w,k}[_{0,\overline{0}}\cap]C_{w,k}[_{m\min(t),\overline{-m\max(t)}}K_p\cap]C_{w,k}[_{\overline{-n\max(t^{-1})},\overline{n\min(t^{-1})}}.K_p\subseteq$$
$$]C_{w,k}[_{m\min(t),\overline{0}}K_p\cap]C_{w,k}[_{0,\overline{n\min(t)}}K_p.$$

The proof of the last point is similar.

□

4. Shimura varieties

Let (G,X) be a Shimura datum. Thus X is a $G(\mathbb{R})$-conjugacy class of homomorphism $h:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m\to G_{\mathbb{R}}$ satisfying a list of familiar axioms ([Del79], section 2.1):

1. For all $h\in X$, the Hodge structure on $\mathfrak{g}_{\mathbb{R}}$ has weight $(1,-1)$, $(0,0)$ and $(-1,1)$.
2. The involution $\mathrm{Ad}(h(i))$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$.
3. The group G^{ad} has no compact factor defined over $\mathbb{Q}$.

Via base change to $\mathbb{C}$, we have $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}}=\mathbb{G}_m\times\mathbb{G}_m$ (given by $z\mapsto(z,\bar{z})$) and projection to the first factor induces a co-character $\mu:\mathbb{G}_m\to G_{\mathbb{C}}$. Associated to μ we have two opposite parabolic subgroups $P^{std}_\mu=\{g\in G_{\mathbb{C}},\lim_{t\to\infty}\mathrm{Ad}(\mu(t))g \text{ exists}\}$ and $P_\mu=\{g\in G_{\mathbb{C}},\lim_{t\to 0}\mathrm{Ad}(\mu(t))g \text{ exists}\}$. We also let M_μ be the Levi quotient of P_μ and P^{std}_μ. We let $FL^{std}_{G,\mu}=G_{\mathbb{C}}/P^{std}_\mu$ and $FL_{G,\mu}=P_\mu\backslash G_{\mathbb{C}}$ be the Flag varieties. Let E be the reflex field, which is the field of definition of the conjugacy class of μ. The two flag varieties are defined over Spec E.

Let $K\subset G(\mathbb{A}_f)$ be a neat compact open subgroup and let $S_K(\mathbb{C})=G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K$ be the corresponding Shimura variety. It has a canonical model $S_K\to$ Spec E [Mil90]. In the rest of this paper, all the compact open subgroups $K\subset G(\mathbb{A}_f)$ are assumed to be neat, so we do not always repeat this assumption.

The most fundamental Shimura data are the Siegel data $(\mathrm{GSp}_{2g},\mathcal{H}_g)$ for all $g\in\mathbb{Z}_{\geq 1}$, where $\mathcal{H}_g$ is the Siegel (upper and lower) half space of matrices $M\in\mathrm{M}_g(\mathbb{C})$ such that ${}^tM=M$ and $\mathrm{Im}(M)$ is definite. The corresponding Shimura varieties parametrize abelian varieties of dimension g, with a polarization and level structure prescribed by K. A Shimura datum (G,X) is of Hodge type if it admits an embedding in a Siegel datum. All PEL Shimura data are of Hodge type. A Shimura datum (G,X) is of abelian type if there exists a datum (G_1,X_1) of Hodge type and a central isogeny $G^{\mathrm{der}}\to G_1^{\mathrm{der}}$ which induces an isomorphism of connected Shimura datum $(G^{\mathrm{ad}},X^+)=(G_1^{\mathrm{ad}},X_1^+)$ where X^+ is is a connected component of X (and similarly X_1^+ is a connected component of X_1).

Example 4.1. Let L be a totally real field extension of $\mathbb{Q}$. The datum $(\mathrm{Res}_{L/\mathbb{Q}}\mathrm{GSp}_{2g},\mathcal{H}_g^{[L:\mathbb{Q}]})$ is of abelian type. We call it a symplectic datum. In the case $g=1$, the corresponding Shimura varieties are the Hilbert modular varieties.

We let $S_K^\star \to \mathrm{Spec}\ E$ be the minimal compactification of S_K. Depending on the auxiliary choice of a projective cone decomposition Σ, we let $S^{tor}_{K,\Sigma} \to \mathrm{Spec}\ E$ be the toroidal compactification of S_K corresponding to Σ. The boundary $D_{K,\Sigma} = S^{tor}_{K,\Sigma} \setminus S_K$ is a Cartier divisor. The cone decompositions Σ are partially ordered by inclusion, and any two cone decompositions admits a common refinement. The cone decompositions Σ which are such that $S^{tor}_{K,\Sigma}$ is a projective smooth scheme are cofinal among all cone decompositions, and we usually choose them this way. We refer to [Pin90] for the construction of these compactifications.

4.1. **Automorphic vector bundles and their cohomology.**

4.1.1. *Automorphic vector bundles.* We choose a field extension F of E which splits G and we work over F in this section. In most of this paper, F will be a finite extension of $\mathbb{Q}_p$, but this is not necessary for the moment. We can pick a representative of μ that is defined over F, and choose a maximal split torus T with $\mu(\mathbb{G}_m) \subset T \subset M_\mu$.

We let $Z_s(G)$ be the largest subtorus of the center $Z(G)$ which is $\mathbb{R}$-split but contains no $\mathbb{Q}$-split subtorus. We let $G^c = G/Z_s(G)$, and define M^c_μ, T^c, P^c_μ, P^c_μ and $P^{c,std}_\mu$ similarly.

Remark 4.2. In the Hodge type case, $Z_s(G) = \{1\}$. For the symplectic datum $(\mathrm{Res}_{L/\mathbb{Q}}\mathrm{GSp}_{2g}, \mathcal{H}_g^{[L:\mathbb{Q}]})$, $Z_s(G)$ is the kernel of the norm $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m$.

Let $\mathrm{Rep}(M^c_\mu)$ be the category of finite dimensional algebraic representations of the reductive group M^c_μ on F-vector spaces.

By [Mil90], thm. 5.1 and [Har89], thm. 4.2 we have a right M^c_μ-torsor M_{dR} over $S^{tor}_{K^pK_p,\Sigma}$ and it corresponds to a functor:

$$\begin{array}{ccc} \mathrm{Rep}(M^c_\mu) & \to & VB(S^{tor}_{K,\Sigma}) \\ V & \mapsto & \mathcal{V}_{K,\Sigma} \end{array}$$

where $VB(S^{tor}_{K,\Sigma})$ is the category of locally free sheaves of finite rank over $S^{tor}_{K,\Sigma}$. This functor is compatible in a natural way with change of level K and of cone decompositions Σ. The locally free sheaves in the essential image of this functor are called automorphic vector bundles. They carry an equivariant action of $G(\mathbb{A}_f)$.

Remark 4.3. We recall the description of $M_{dR} \times_{S^{tor}_{K,\Sigma}} S_K(\mathbb{C})$ as a complex analytic space. First, we have the Borel embedding $\beta : X \hookrightarrow FL^{std}_{G,\mu}(\mathbb{C}) = G^c(\mathbb{C})/P^{c,std}_\mu(\mathbb{C})$ which sends $h \in X$ to the parabolic stabilizing the Hodge filtration. The Borel embedding is equivariant for the left action of $G(\mathbb{R})$. We have a the canonical map $G^c(\mathbb{C}) \to FL^{std}_{G,\mu}(\mathbb{C})$. This map is a right $P^{c,std}_\mu(\mathbb{C})$-torsor and is $G(\mathbb{C})$-equivariant for the left action. Similarly, the canonical map $G^c(\mathbb{C})/U_{P^{c,std}_\mu}(\mathbb{C}) \to FL^{std}_{G,\mu}(\mathbb{C})$ is a right $M^c_\mu(\mathbb{C})$-torsor. Then we have:

$$M_{dR} \times_{S^{tor}_{K,\Sigma}} S_K(\mathbb{C}) = G(\mathbb{Q})\backslash\big(\beta^{-1}(G^c(\mathbb{C})/U_{P^{c,std}_\mu}(\mathbb{C})) \times G(\mathbb{A}_f)/K.$$

Remark 4.4. We give the description of M_{dR} for the Siegel Shimura datum $(\mathrm{GSp}_{2g}, \mathcal{H}_g)$. Let us first work over S_K. Let ω_A be the co-normal sheaf of the universal abelian scheme $A \to S_K$ along its unit section and $\mathrm{Lie}(A)$ be its dual. The torsor M_{dR} parametrizes trivializations $\psi_1 \oplus \psi_2 : \mathscr{O}^g_{S_K} \oplus \mathscr{O}^g_{S_K} \to \mathrm{Lie}(A) \oplus \omega_{A^t}$, such that under the isomorphism $\mathrm{Lie}(A)^\vee = \omega_{A^t}$ given by the polarization, we have $\psi_1 = c(\psi_2^{-1})^t$ for a unit $c \in \mathscr{O}^\times_{S_K}$. The torsor M_{dR} extends to a Zariski torsor that we keep denoting M_{dR} over $S^{tor}_{K,\Sigma}$. Indeed, A can be extended to a semi-abelian scheme A_Σ over $S^{tor}_{K,\Sigma}$, and we can let M_{dR} be the torsor of trivializations of $\mathrm{Lie}(A_\Sigma) \oplus \omega_{A^t_\Sigma}$, compatible, up to a unit, with the polarization.

We now make the construction of automorphic vector bundles explicit, and label them using weights. We begin by making a choice of positive roots of T: we first choose a set of compact positive roots Φ^+_c which lie in $\mathfrak{m}_\mu$ the Lie algebra of M_μ. We then choose the non-compact positive roots Φ^+_{nc} to lie in $\mathfrak{g}/\mathfrak{p}^{std}_\mu$ where $\mathfrak{p}^{std}_\mu$ is the Lie algebra of P^{std}_μ. We let $\Phi^+ = \Phi^+_c \coprod \Phi^+_{nc}$.

Remark 4.5. This choice implies that the Borel corresponding to Φ^+ is included in P_μ, or equivalently that the cocharacter μ of the Shimura datum is dominant. In section 3 we fixed a parabolic P of G containing a Borel B. In the applications to Shimura varieties, P will be P_μ. Therefore our convention is also compatible with the choice made in section 3.

We let $X^\star(T)^{M_\mu,+}$ be the cone of characters of T which are dominant for Φ^+_c. We label irreducible representations of M_μ by their highest weight $\kappa \in X^\star(T)^{M_\mu,+}$. An explicit construction of the highest weight κ representation V_κ is as follows. Let $w_{0,M}$ be the longest element of the Weyl group of M_μ. For any $\kappa \in X^\star(T)^{M_\mu,+}$ we consider the space V_κ of functions $f : M_\mu \to \mathbb{A}^1$ such that $f(mb) = (w_{0,M}\kappa)(b^{-1})f(m)$ for all $m \in M_\mu$ and $b \in B \cap M_\mu$. The action of M_μ on itself via left translation induces a left action on V_κ, i.e. $(m' \cdot f)(m) = f(m'^{-1}(m))$. The irreducible representations of M^c_μ are the irreducible representations of M_μ labelled by dominant characters κ of T^c. We let $X^\star(T^c)^{M_\mu,+}$ be the cone of these characters.

We denote by $\mathcal{V}_{\kappa,K,\Sigma}$ the locally free sheaf associated to the irreducible representation of highest weight κ of M^c_μ. Concretely, we consider the right torsor $g : M_{dR} \to S^{tor}_{K,\Sigma}$ and we let $\mathcal{V}_{\kappa,K,\Sigma}$ be the subsheaf of $g_\star\mathscr{O}_{M_{dR}}$ of sections $f(m)$ such that $f(mb) = w_{0,M}\kappa(b^{-1})f(m)$ for all $b \in B \cap M_\mu$. We will often abbreviate $\mathcal{V}_{\kappa,K,\Sigma}$ to $\mathcal{V}_\kappa$.

We also introduce the subsheaf $\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})$ where $D_{K,\Sigma} \hookrightarrow S^{tor}_{K,\Sigma}$ is the Cartier divisor of the boundary $S^{tor}_{K,\Sigma} \setminus S_{K,\Sigma}$. Again, we often abbreviate this sheaf to $\mathcal{V}_\kappa(-D)$.

4.1.2. *The cohomology of automorphic vector bundles.* We let $\pi_{K,\Sigma} : S^{tor}_{K,\Sigma} \to S^\star_K$ be the projection from toroidal to minimal compactification.

Theorem 4.6. *We have* $\mathrm{R}^i(\pi_{K,\Sigma})_\star\mathcal{V}_{\kappa,K,\Sigma}(-nD_{K,\Sigma}) = 0$ *for all* $i > 0$ *and all* $n \geq 1$.

Proof. In the PEL case (and for $n = 1$), this is [Lan17], thm. 8.6. We give an argument which follows closely [AIP15] and which is also similar to *loc cit.* Let $x \in S^\star_K$. We write $\widehat{S^{tor}_{K,\Sigma}}^x$ for the formal completion of $S^{tor}_{K,\Sigma}$ along $\pi^{-1}_{K,\Sigma}(x)$. By the theorem on formal functions ([Sta13], Tag O207) It suffices to prove that $\mathrm{H}^i(\widehat{S^{tor}_{K,\Sigma}}^x, \mathcal{V}_{\kappa,K,\Sigma}(-nD_{K,\Sigma})) = 0$ for all $i > 0$ and $n \geq 1$.

We may now use the description of $\widehat{S^{tor}_{K,\Sigma}}^x$ in terms of the local charts, following the notations of [MP19]. The argument is mostly about torus embedding, so we do not need to explain in detail the structure of the toroidal compactification, but just recall what is strictly necessary. Suppose that x belongs to a boundary component indexed by a cusp label representative Φ. There is a tower of spaces:

$$\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \overset{g}{\to} \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi) \to \mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$$

where $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi)$ is a torsor under a torus $\mathbf{E}_K(\Phi)$. There is a locally free sheaf $\mathcal{V}_{\kappa,K}$ over $\mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi)$. There is a twisted torus embedding $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$ which depends on the choice of Σ. There is an arithmetic group $\Delta_K(\Phi)$ acting on $X^\star(\mathbf{E}_K(\Phi))$ and on $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \hookrightarrow \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$. Let $\overline{g} : \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi)$. The arithmetic group also acts on $\overline{g}^\star\mathcal{V}_{\kappa,K}$. We have a $\Delta_K(\Phi)$-invariant closed subscheme $\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \hookrightarrow \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$. There is a finite morphism $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}) \to \mathbf{S}_K^\star$ whose image contains x. There is a series of morphisms:

$$\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi) \to \mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}).$$

We let $\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))_x$ be the closed subspace equal to the inverse image of x. The main result on the description of toroidal compactifications states that:

$$\widehat{S^{tor}_{K,\Sigma}}^x \simeq \Delta_K(\Phi)\backslash\big(\mathbf{S}_{K_\Phi}(Q\widehat{_\Phi, D}_\Phi, \Sigma(\Phi))^{\mathbf{Z}_{K_\Phi}(Q_\Phi,D_\Phi,\Sigma(\Phi))_x}\big)$$

Moreover, the sheaf $\overline{g}^\star\mathcal{V}_{\kappa,K}$ on $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$ descends to

$$\Delta_K(\Phi)\backslash\big(\mathbf{S}_{K_\Phi}(Q\widehat{_\Phi, D}_\Phi, \Sigma(\Phi))^{\mathbf{Z}_{K_\Phi}(Q_\Phi,D_\Phi,\Sigma(\Phi))_x}\big)$$

and identifies with $\mathcal{V}_{\kappa,K,\Sigma}|_{\widehat{S^{tor}_{K,\Sigma}}^x}$. We let D be the boundary divisor $S^{tor}_{K,\Sigma}\setminus S_K$. Under the isomorphism above, it corresponds to the divisor $D_0 = \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))\setminus \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi)$.

There is a divisor D' on $S^{tor}_{K,\Sigma}$ which has exactly the same support as D and such that $\mathscr{O}_{S^{tor}_{K,\Sigma}}(-D')$ is ample relatively to the minimal compactification. It corresponds to a divisor D'_0 on $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$.

We will prove the following statement: for any $C \in \mathbb{Z}_{\geq 0}$, there exists $s \geq C$, a finite morphism

$$\psi : \widehat{S^{tor}_{K,\Sigma}}^x \to \widehat{S^{tor}_{K,\Sigma}}^x$$

such that $\psi^\star\mathcal{V}_{\kappa,K,\Sigma} = \mathcal{V}_{\kappa,K,\Sigma}$ and we have a split injection $\mathscr{O}(-nD) \to \psi_\star\mathscr{O}(-sD')$. This implies the theorem, as we deduce that for all $i \geq 0$, $\mathrm{H}^i(\widehat{S^{tor}_{K,\Sigma}}^x, \mathcal{V}_{\kappa,K,\Sigma}(-nD))$ is a direct factor of $\mathrm{H}^i(\widehat{S^{tor}_{K,\Sigma}}^x, \mathcal{V}_{\kappa,K,\Sigma}(-sD'))$. Taking C large enough, this last group vanishes. We now prove the claim about the existence of ψ. Our proof follows [AIP15], p. 679. We will construct everything on $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$. For any integer ℓ, the multiplication by ℓ-map on the torus $\mathbf{E}_K(\Phi)$ induces a finite morphism $\psi_\ell : \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \to \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$ which is $\Delta_K(\Phi)$ equivariant and for which $\psi_\ell^\star\overline{g}^\star\mathcal{V}_{\kappa,K} = \overline{g}^\star\mathcal{V}_{\kappa,K}$. The morphisms ψ_ℓ induces a finite morphism of $\widehat{S^{tor}_{K,\Sigma}}^x$. We have $D'_0 = \sum_{\sigma\in\Sigma(\Phi)(1)} a_\rho D_\rho$ where $\Sigma(\Phi)(1)$ is the set of one dimensional faces in $\Sigma(\Phi)$ and $D_0 = \sum_{\sigma\in\Sigma(\Phi)(1)} D_\rho$. Since D'_0 is $\Delta_K(\Phi)$-equivariant and $\Sigma(\Phi)(1)/\Delta_K(\Phi)$ is finite, we deduce that for any $s \geq 1$, there exists ℓ such that $0 < sa_\rho \leq \ell$. It follows that the round-down of the $\mathbb{Q}$-divisor $-\ell^{-1}D'_0$ is $-D_0$. We deduce from [CLS11], lem. 9.3.4 that $\mathscr{O}(-D_0) \hookrightarrow (\psi_\ell)_\star\mathscr{O}(-sD'_0)$ is a split injection. Pulling back this morphism by ψ_n, we deduce that $\mathscr{O}(-nD_0) \hookrightarrow (\psi_\ell)_\star\mathscr{O}(-snD'_0)$ is a split injection. □

We let $\pi_{K,K',\Sigma,\Sigma'} : S^{tor}_{K,\Sigma} \to S^{tor}_{K',\Sigma'}$ be the map associated to a change of level (for $K' \subseteq K$) and cone decomposition.

Theorem 4.7. *We have* $\mathrm{R}^i(\pi_{K,K',\Sigma,\Sigma'})_\star \mathcal{V}_{\kappa,K,\Sigma} = 0$ *and* $\mathrm{R}^i(\pi_{K,K',\Sigma,\Sigma'})_\star \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}) = 0$ *for all* $i > 0$*. Moreover if* $K' \subseteq K$ *is normal then we have*

$$(\mathrm{R}(\pi_{K,K',\Sigma,\Sigma'})_\star \mathcal{V}_{\kappa,K,\Sigma})^{K/K'} = \mathcal{V}_{\kappa,K',\Sigma'}$$

and

$$(\mathrm{R}(\pi_{K,K',\Sigma,\Sigma'})_\star \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))^{K/K'} = \mathcal{V}_{\kappa,K',\Sigma'}(-D_{K',\Sigma'}).$$

Proof. This is easily extracted from [Har90b], section 2 (look in particular at proposition 2.4 and proposition 2.6 and their proofs). In the PEL case, this is explicitely [Lan17], prop. 7.5. □

In particular from the case that $K = K'$ we deduce that the cohomologies $\mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma})$ and $\mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))$ do not depend on Σ.

We also recall that the maps $\pi_{K,K',\Sigma,\Sigma'}$ have fundamental classes in the sense of [FP19] section 2.3. Namely we have maps $\mathscr{O}_{S^{tor}_{K,\Sigma}} \to \pi^!_{K,K',\Sigma,\Sigma}\mathscr{O}_{S^{tor}_{K',\Sigma'}}$ and $\mathscr{O}_{S^{tor}_{K,\Sigma}}(-D_{K,\Sigma}) \to \pi^!_{K,K',\Sigma,\Sigma}\mathscr{O}_{S^{tor}_{K',\Sigma'}}(-D_{K',\Sigma'})$ or equivalently by adjunction trace maps $\mathrm{R}(\pi_{K,K',\Sigma,\Sigma'})_\star \mathscr{O}_{S^{tor}_{K,\Sigma}} = (\pi_{K,K',\Sigma,\Sigma'})_\star \mathscr{O}_{S^{tor}_{K,\Sigma}} \to \mathscr{O}_{S^{tor}_{K',\Sigma'}}$ and $\mathrm{R}(\pi_{K,K',\Sigma,\Sigma'})_\star(\mathscr{O}_{S^{tor}_{K,\Sigma}}(-D_{K,\Sigma})) = (\pi_{K,K',\Sigma,\Sigma'})_\star(\mathscr{O}_{S^{tor}_{K,\Sigma}}(-D_{K,\Sigma})) \to \mathscr{O}_{S^{tor}_{K',\Sigma'}}(-D_{K',\Sigma'})$, both of which extend the trace map for the finite étale morphism $\pi_{K,K'} : S_K \to S_{K'}$.

4.2. **Action of the Hecke algebra.**

4.2.1. *Action of Hecke correspondences on cohomology.* Let $K_1, K_2 \subset G(\mathbb{A}_f)$ be open compact subgroups, and let $g \in G(\mathbb{A}_f)$. To this data we associate a Hecke correspondence

$$\begin{array}{ccccc} & & S_{K_1\cap gK_2g^{-1}} & & \\ & \swarrow^{p_2} & & \searrow^{p_1} & \\ S_{K_2} & & & & S_{K_1} \end{array}$$

where p_1 is the forgetful map corresponding to the inclusion $K_1 \cap gK_2g^{-1} \subseteq K_1$, and p_2 is the composition of the action map $[g] : S_{K_1\cap gK_2g^{-1}} \to S_{g^{-1}K_1g\cap K_2}$ and the forgetful map corresponding to the inclusion $g^{-1}K_1g \cap K_2 \subseteq K_2$.

For any weight $\kappa \in X^\star(T^c)^{M_\mu,+}$ we have a cohomological correspondence $(p_1)_\star p_2^\star \mathcal{V}_{\kappa,K_2} \to \mathcal{V}_{\kappa,K_1}$. This is obtained by combining the trace map $\mathrm{tr}_{p_1} : (p_1)_\star \mathcal{O}_{S_{K_1\cap gK_2g^{-1}}} \to \mathcal{O}_{S_{K_1}}$ for the finite étale map p_1 with the isomorphism $p_2^\star \mathcal{V}_{\kappa,K_2} \simeq p_1^\star \mathcal{V}_{\kappa,K_1}$, which is itself composed of the isomorphism $p_1^\star \mathcal{V}_{\kappa,K_1} \simeq \mathcal{V}_{\kappa,K_1\cap gK_2g^{-1}}$ and the similar isomorphism for the forgetful map $S_{g^{-1}K_1g\cap K_2} \to S_{K_2}$, as well as the action map $[g]^\star \mathcal{V}_{\kappa,g^{-1}K_1g\cap K_2} \simeq \mathcal{V}_{\kappa,K_1\cap gK_2g^{-1}}$.

One readily verifies that the cohomological correspondence $((S_{K_1\cap gK_2g^{-1}}, p_1, p_2), (p_1)_\star p_2^\star \mathcal{V}_{\kappa,K_2} \to \mathcal{V}_{\kappa,K_1})$ only depends on the double coset K_1gK_2, up to canonical isomorphism.

Now for suitable choices of cone decomposition we have a diagram

$$\begin{array}{ccccc} & & S^{tor}_{K_1\cap gK_2g^{-1},\Sigma''} & & \\ & \swarrow^{p_2} & & \searrow^{p_1} & \\ S^{tor}_{K_2,\Sigma'} & & & & S^{tor}_{K_1,\Sigma} \end{array}$$

and we claim that our cohomological correspondence on the interior extends to a cohomological correspondence $\mathrm{R}(p_1)_\star p_2^\star \mathcal{V}_{\kappa,K_2,\Sigma'} = (p_1)_\star p_2^\star \mathcal{V}_{\kappa,K_2,\Sigma'} \to \mathcal{V}_{\kappa,K_1,\Sigma''}$ as well as a cuspidal version $\mathrm{R}(p_1)_\star p_2^\star \mathcal{V}_{\kappa,K_2,\Sigma'}(-D_{K_2,\Sigma'}) = (p_1)_\star p_2^\star \mathcal{V}_{\kappa,K_2,\Sigma'}(-D_{K_2,\Sigma'}) \to \mathcal{V}_{\kappa,K_1,\Sigma''}(-D_{K_1,\Sigma})$. We have already discussed the extensions of the trace map in the previous section. As for the action map, it also induces an isomorphism of canonical extensions $p_2^\star \mathcal{V}_{\kappa,K_2,\Sigma} \to p_1^\star \mathcal{V}_{\kappa,K_1,\Sigma}$ as well as a morphism of subsheaves $p_2^\star(\mathcal{V}_{\kappa,K_2,\Sigma}(-D_{K_2,\Sigma'})) \to (p_1^\star \mathcal{V}_{\kappa,K_1,\Sigma})(-D_{K_1\cap gK_2g^{-1},\Sigma''})$.

Finally these cohomological correspondences induces maps on cohomology in the usual way, namely we denote by $[K_1gK_2]$ the composition

$$\mathrm{R}\Gamma(S^{tor}_{K_2,\Sigma'}, \mathcal{V}_{\kappa,K_2,\Sigma'}) \to \mathrm{R}\Gamma(S^{tor}_{K_1\cap gK_2g^{-1},\Sigma''}, p_2^\star \mathcal{V}_{\kappa,K_2,\Sigma'}) =$$

$$\mathrm{R}\Gamma(S^{tor}_{K_1,\Sigma}, \mathrm{R}(p_1)_* p_2^* \mathcal{V}_{\kappa,K_2,\Sigma'}) \to \Gamma(S^{tor}_{K_1,\Sigma}, \mathcal{V}_{\kappa,K_1,\Sigma})$$

where the first map is p_2^* and the last map uses the cohomological correspondence. We have a similar definition for cuspidal cohomology.

4.2.2. *Composition of Hecke correspondences.* Let us briefly explain the point of this section. We have defined an action of individual Hecke operators on coherent cohomology, and we would like to show that this actually gives an action of the Hecke algebra. The standard proof (see [Har90b, Prop. 2.6]) is to pass to a limit over all K to obtain a representation of $G(\mathbb{A}_f)$, and then use the purely group theoretic fact that Hecke algebras act on its invariants. However in this paper we will also study coherent cohomology with support conditions which are not $G(\mathbb{Q}_p)$ invariant, and so we no longer expect an action of the full Hecke algebra, but only of certain Hecke operators which preserve the support conditions in a suitable sense. We would still like to know that these Hecke operators compose according to relations in the abstract Hecke algebra.

For this reason we develop a different approach to composition of Hecke operators, by directly studying the geometric composition of Hecke correspondences. This material is presumably well known but we lack a reference (see [FC90, VII §3] for a closely related discussion.)

We recall the formalism of double coset multiplication. Fix a Haar measure on $G(\mathbb{A}_f)$ and let $C_c^\infty(G(\mathbb{A}_f),\mathbb{R})$ be the Hecke algebra with its convolution product. For $K_1, K_2 \subset G(\mathbb{A}_f)$ we consider the free group $\mathbb{Z}[K_1\backslash G(\mathbb{A}_f)/K_2]$ and we write the basis elements as $[K_1gK_2]$. We have an embedding $i_{K_1,K_2} : \mathbb{Z}[K_1\backslash G(\mathbb{A}_f)/K_2] \to C_c^\infty(G(\mathbb{A}_f),\mathbb{R})$ which sends $[K_1gK_2]$ to $\frac{1}{\sqrt{\mathrm{vol}(K_1)\mathrm{vol}(K_2)}}\mathbf{1}_{K_1gK_2}$, where $\mathbf{1}_{K_1gK_2}$ denotes the characteristic function of the double coset K_1gK_2.

For $K_1, K_2, K_3 \subset G(\mathbb{A}_f)$ open compact, there is a product map

$$\mathbb{Z}[K_1\backslash G(\mathbb{A}_f)/K_2] \times \mathbb{Z}[K_1\backslash G(\mathbb{A}_f)/K_2] \to \mathbb{Z}[K_1\backslash G(\mathbb{A}_f)/K_3]$$

which can be defined by

$$i_{K_1,K_3}([K_1gK_2][K_2hK_3]) = i_{K_1,K_2}([K_1gK_2]) \star i_{K_2,K_3}([K_2hK_3])$$

To see that the right hand side is in the image of i_{K_1,K_3} note that

$$(\mathbf{1}_{K_1gK_2} \star \mathbf{1}_{K_2hK_3})(x) = \mathrm{vol}(K_1gK_2 \cap xK_3h^{-1}K_2)$$

is an integer multiple of $\mathrm{vol}(K_2)$. We also note that this definition is independent of the choice of Haar measure.

It follows from the definition that double coset multiplication is associative and satisfies

$$([K_1gK_2][K_2hK_3])^t = [K_2hK_3]^t[K_1gK_2]^t$$

where the transpose map $(-)^t : \mathbb{Z}[K_1\backslash G(\mathbb{A}_f)/K_2] \to \mathbb{Z}[K_2\backslash G(\mathbb{A}_f)/K_1]$ is defined by $[K_1gK_2]^t = [K_2g^{-1}K_1]$. This corresponds to transposes of Hecke correspondences.

We now give a formula for double coset multiplication which is closely related to geometric composition of Hecke correspondences.

Proposition 4.8. *Let $k_1, \ldots, k_n \in K_2$ be a set of representatives for the double cosets $(K_2 \cap g^{-1}K_1g)\backslash K_2/(K_2 \cap hK_3h^{-1})$. Then we have*

$$[K_1gK_2][K_2hK_3] = \sum_{i=1}^{n} c(k_i)[K_1gk_ihK_3]$$

where $c(k_i) = [g^{-1}K_1g \cap (k_ih)K_3(k_ih)^{-1} : g^{-1}K_1g \cap K_2 \cap (k_ih)K_3(k_ih)^{-1}]$

Proof. We first note that for $x, y \in G(\mathbb{A}_f)$ we have $\mathbf{1}_{K_1x} \star \mathbf{1}_{yK_3} = \mathrm{vol}(x^{-1}K_1x \cap yK_3y^{-1})\mathbf{1}_{K_1xyK_3}$. Decomposing K_1gK_2 into right K_1 cosets and K_2hK_3 into left K_3 cosets we have

$$\mathbf{1}_{K_1gK_2} \star \mathbf{1}_{K_2hK_3} = \sum_{\substack{x\in K_1\backslash K_1gK_2 \\ y\in K_2hK_3/K_3}} \mathrm{vol}(x^{-1}K_1x \cap yK_3y^{-1})\mathbf{1}_{K_1xyK_3}.$$

Now note that the terms of this sum are constant in K_2 orbits, for the action $k \cdot (K_1x, yK_3) = (K_1xk^{-1}, kyK_3)$. We have a bijection

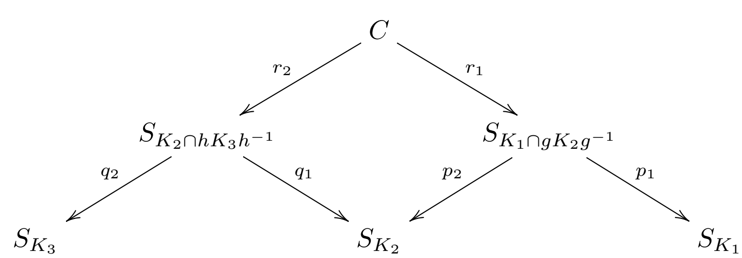
$$(K_2 \cap g^{-1}K_1g)\backslash K_2/(K_2 \cap hK_3h^{-1}) \to K_2\backslash((K_1\backslash K_1gK_2) \times (K_2hK_3/K_3))$$

sending the double coset of k to the orbit of (K_1g, khK_3). Moreover the stabilizer of this orbit is $K_2 \cap g^{-1}K_1g \cap (kh)K_3(kh)^{-1}$. It follows that

$$\mathbf{1}_{K_1gK_2} \star \mathbf{1}_{K_2hK_3} = \sum_{i=1}^{n} \frac{\mathrm{vol}(K_2)\mathrm{vol}(g^{-1}K_1g \cap (k_ih)K_3(k_ih)^{-1})}{\mathrm{vol}(K_2 \cap g^{-1}K_1g \cap (k_ih)K_3(k_ih)^{-1})}\mathbf{1}_{K_1gk_ihK_3}.$$

which translated back into double coset multiplication gives the proposition. □

Suppose we have $K_1, K_2, K_3 \subset G(\mathbb{A}_f)$ open compact subgroups and $g, h \in G(\mathbb{A}_f)$. We have a diagram of correspondences

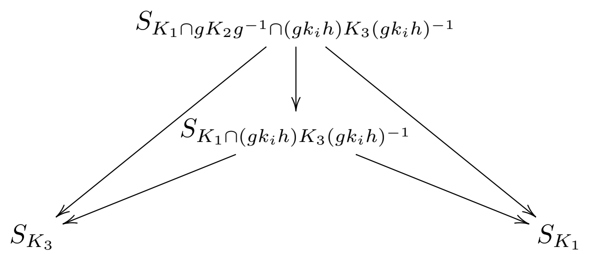


where the middle diamond is cartesian. We denote $s_1 = p_1r_1$, $s_2 = q_2r_2$.

It is not true in general that the correspondence (C, s_1, s_2) is isomorphic to a disjoint union of Hecke correspondences. However this is almost the case in a way we now explain.

We first construct another correspondence (C', s'_1, s'_2) between S_{K_1} and S_{K_3}. Let $k_1, \ldots, k_n$ be as in proposition 4.8. Now let $C' = \coprod_i S_{K_1\cap gK_2g^{-1}\cap(gk_ih)K_3(gk_ih)^{-1}}$, let $s'_1 : C' \to S_{K_1}$ be the forgetful map on each component, and let $s'_2 : C' \to S_{K_3}$

be the action map $[gk_ih]$ followed by the forgetful map. Then we note that while the components of C' are not necessarily Hecke correspondences themselves, we nonetheless have commutative diagrams



where the vertical arrow and rightward arrows are forgetful maps and the leftward arrows are the action maps for $[gk_ih]$ followed by forgetful maps. In particular the bottom correspondence is exactly the Hecke correspondence corresponding to the double coset $K_1gk_ihK_3$.

Proposition 4.9. *There is an isomorphism of correspondences* $(C, s_1, s_2) \simeq (C', s'_1, s'_2)$.

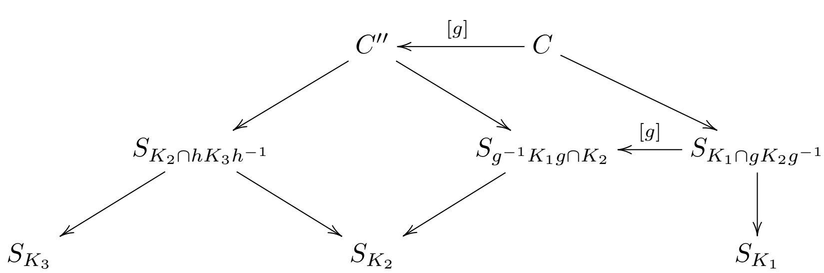
We will use the following group theoretic lemma.

Lemma 4.10. *Let K be a group and $H_1, H_2 \subseteq K$ subgroups. Let X be a right K-torsor. Then there is a bijection*

$$\begin{aligned} \coprod_{H_1gH_2\in H_1\backslash K/H_2} X/gH_1g^{-1}\cap H_2 &\to X/H_1 \times X/H_2 \\ x(gH_1g^{-1}\cap H_2) &\mapsto (xgH_1, xH_2) \end{aligned}$$

Proof of proposition 4.9. We recall that a point of the Shimura variety S_K over a connected locally noetherian base is some data which is independent of K (an isogeny class of abelian varieties with Hodge tensors) to which one associates a right $G(\mathbb{A}_f)$-torsor, and a K level structure is just a K orbit in this $G(\mathbb{A}_f)$-torsor. Moreover for any $K' \subset K$ the forgetful map $S_{K'} \to S_K$ at the level of points sends the K'-orbit to the K-orbit generated by it, and the action map $[g] : S_K \to S_{g^{-1}Kg}$ is given by right multiplication by g.

With this observation the proof of the proposition proceeds formally. We can consider a slightly expanded diagram.



Here all three quadrilaterals are Cartesian. We can compute C'' using lemma 4.10. Indeed giving a point of C'' lying over a point of S_{K_2} is the same as giving a pair of an $H_1 = K_2 \cap hK_3h^{-1}$ and $H_2 = g^{-1}K_1g \cap K_2$ orbit inside a $K = K_2$-torsor X.

We deduce from lemma 4.10 an isomorphism

$$C'' \simeq \coprod_{i=1}^{n} S_{g^{-1}K_1g\cap K_2\cap (k_ih)K_3(k_ih)^{-1}}$$

where on each factor, the map to $S_{g^{-1}K_1g\cap K_2}$ is the forgetful map, while the map to $S_{K_2\cap hK_3h^{-1}}$ is the action map $[k_i]$ followed by the forgetful map. Then the isomorphism $C \simeq C'$ is obtained as the composition

$$C \overset{[g]}{\to} C'' \to \coprod_{i=1}^{n} S_{g^{-1}K_1g\cap K_2\cap (k_ih)K_3(k_ih)^{-1}} \overset{[g]^{-1}}{\to} C'.$$

Moreover from the descriptions of the maps out of C'' above one immediately deduces the compatibility between s_1, s_1' and s_2, s_2'. □

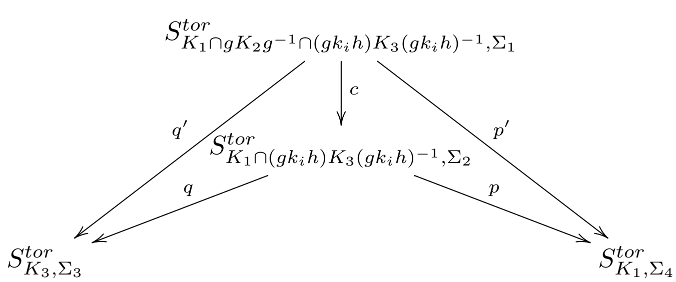
The following proposition is readily deduced from [Har90b, Prop. 2.6], but we give an alternative proof which will also work for cohomology with support.

Proposition 4.11. *We have* $[K_1gK_2]\circ[K_2hK_3] = [K_1gK_2][K_2hK_3]$ *as maps*

$$\mathrm{R}\Gamma(S^{tor}_{K_3,\Sigma''}, \mathcal{V}_{\kappa,K_3,\Sigma''}) \to \mathrm{R}\Gamma(S^{tor}_{K_1,\Sigma}, \mathcal{V}_{\kappa,K_1,\Sigma}).$$

Proof. We can choose toroidal compactifications of all the Shimura varieties occurring in this section: S_{K_i}, $i=1,2,3$, $S_{K_1\cap gK_2g^{-1}}$,$S_{K_2\cap hK_3h^{-1}}$, $S_{K_1\cap(gk_ih)K_3(gk_ih)^{-1}}$, $S_{K_1\cap gK_2g^{-1}\cap(gk_ih)K_3(gk_ih)^{-1}}$ so that all the diagrams appearing above extend to the compactifications (we will not be particularly consistant or careful with our labelling of the various cone decompositions in this argument, and they will be denoted by Σ_* where $*$ is an index). In particular we obtain a compactification C^{tor} of the correspondence C which is isomorphic to a disjoint union of toroidal compactifications of Shimura varieties. Moreover using this description, we can construct cohomological correspondences over C^{tor} as in section 4.2.1 using the action maps for gk_ih. We first claim that this cohomological correspondence acts on cohomology by the linear combination of Hecke operators on the right hand side of the formula in proposition 4.8.

Consider a commutative diagram



We claim that the following diagram commutes (where the vertical map is induced by the adjunction $\mathrm{Id}\Rightarrow c_\star c^\star$, the other maps are given by the cohomological correspondence, and $c(k_i)$ is the generic degree of the generically finite flat map c):

$$\begin{array}{ccc} \mathrm{R}p'_\star(q')^\star\mathcal{V}_{\kappa,K_3,\Sigma_3} & \xrightarrow{tr_{p'}} & \mathcal{V}_{\kappa,K_1,\Sigma_1} \\ \uparrow & \nearrow_{c(k_i)tr_p} & \\ \mathrm{R}p_\star(q)^\star\mathcal{V}_{\kappa,K_3,\Sigma_3} & & \end{array}$$

These are maps of locally free sheaves by theorem 4.7, and the commutativity of the diagram follows from its commutativity away from the boundary which is clear.

Hence to prove the proposition, we need to see that the action of the cohomological correspondence C^{tor} is equal to $[K_1gK_2]\circ[K_2hK_3]$.

Consider the diamond from the proof of proposition 4.9

$$\begin{array}{ccccc} & & C''^{tor} & & \\ & \swarrow^{p'} & & \searrow^{q'} & \\ S^{tor}_{K_2\cap hK_3h^{-1},\Sigma_a} & & & & S^{tor}_{g^{-1}K_1g\cap K_2,\Sigma_b} \\ & \searrow_{q} & & \swarrow_{p} & \\ & & S^{tor}_{K_2,\Sigma'} & & \end{array}$$

which is cartesian away from the boundary. We would like to know that

$$\begin{array}{ccccc} & & \mathrm{R}\Gamma(C''^{tor},(q')^\star\mathcal{V}_{\kappa,g^{-1}K_1g\cap K_2,\Sigma_b}) & & \\ & \nearrow^{(p')^\star} & & \searrow^{\mathrm{tr}_{q'}} & \\ \mathrm{R}\Gamma(S^{tor}_{K_2\cap hK_3h^{-1},\Sigma_a},\mathcal{V}_{\kappa,K_2\cap hK_3h^{-1},\Sigma_a}) & & & & \mathrm{R}\Gamma(S^{tor}_{g^{-1}K_1g\cap K_2,\Sigma_b},\mathcal{V}_{\kappa,g^{-1}K_1g\cap K_2,\Sigma_b}) \\ & \searrow_{\mathrm{tr}_q} & & \nearrow_{p^\star} & \\ & & \mathrm{R}\Gamma(S^{tor}_{K_2,\Sigma'},\mathcal{V}_{\kappa,K_2,\Sigma'}) & & \end{array}$$

commutes, and similarly for cuspidal cohomology. Note that here what we write as $(p')^\star$ is really the composition

$$\mathrm{R}\Gamma(S^{tor}_{K_2\cap hK_3h^{-1},\Sigma_a},\mathcal{V}_{\kappa,K_2\cap hK_3h^{-1},\Sigma_a})\to\mathrm{R}\Gamma(C''^{tor},(p')^\star\mathcal{V}_{\kappa,K_2\cap hK_3h^{-1},\Sigma_a})\to$$
$$\mathrm{R}\Gamma(C''^{tor},(q')^\star\mathcal{V}_{\kappa,g^{-1}K_1g\cap K_2,\Sigma_b})$$

where the first map is literally $(p')^\star$ and the second map is described on each component of C''^{tor} as the action map $[k_i]$ (see the description of p',q' given in the proof of Proposition 4.9.)

We claim that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{R}(q')_\star(p')^\star q^\star\mathcal{V}_{\kappa,K_2\cap hK_3h^{-1},\Sigma_a}=\mathrm{R}(q')_\star(q')^\star p^\star\mathcal{V}_{\kappa,K_2\cap hK_3h^{-1},\Sigma_a} & \longrightarrow & p^\star\mathcal{V}_{\kappa,g^{-1}K_1g\cap K_2,\Sigma_b} \\ \uparrow & \nearrow & \\ \mathrm{R}p^\star q_\star q^\star\mathcal{V}_{\kappa K_2\cap hK_3h^{-1},\Sigma_a} & & \end{array}$$

This is a diagram of locally free sheaves by theorem 4.7, and therefore the commutativity can be checked away from the boundary, where this is clear. $\square$

Remark 4.12. We remark that for $K' \subseteq K \subset G(\mathbb{A}_f)$ open compact subgroups, the Hecke operator $[K'1K]$ and $[K1K']$ are the pullback and trace for the forgetful map $S_{K',\Sigma'} \to S_{K,\Sigma}$. Moreover if $g \in G(\mathbb{A}_f)$ then $[Kg(g^{-1}Kg)]$ is simply the action map $[g]^\star$. Finally for $K_1, K_2 \subset G(\mathbb{A}_f)$ open compact subgroups and $g \in G(\mathbb{A}_f)$, we have a factorization

$$[K_1gK_2] = [K_1 1(K_1 \cap gK_2g^{-1})][(K_1 \cap gK_2g^{-1})g(g^{-1}K_1g \cap K_2)][(g^{-1}K_1g \cap K_2)1K_2]$$

of $[K_1gK_2]$ into a pullback map, an action map, and a trace, which is essentially the definition of $[K_1gK_2]$.

4.2.3. *Serre duality.* The dualizing sheaf of $S_{K,\Sigma}$ is $\mathcal{V}_{-2\rho_{nc},K,\Sigma}(-D_{K,\Sigma})$ where ρ_{nc} is half the sum of the non-compact positive roots. Indeed, recall our choice of non-compact positive roots being in $\mathfrak{g}/\mathfrak{p}_\mu^{std}$ and that $\mathfrak{g}/\mathfrak{p}_\mu^{std}$ is the tangent space at the identity of $FL_{G,\mu}^{std}$ and then apply [Har90b], proposition 2.2.6.

The Serre dual of the automorphic sheaf $\mathcal{V}_{\kappa,K,\Sigma}$ is therefore $\mathcal{V}_{-2\rho_{nc}-w_{0,M}\kappa,K,\Sigma}(-D_{K,\Sigma})$ where $w_{0,M}$ is the longest element of the Weyl group of M_μ. Serre duality is:

Proposition 4.13. *There is a Serre duality isomorphism*

$$D_F(\mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}))[-d] \simeq \mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \mathcal{V}_{-2\rho_{nc}-w_{0,M_\mu}\kappa,K,\Sigma}(-D_{K,\Sigma}))$$

where $D_F(-) = \mathrm{RHom}_F(-, F)$ *is the dualizing functor for* F*-vector spaces and* d *is the dimension of* S_K*. Moreover this isomorphism, is compatible with the Hecke action in the sense the action of* $[KgK]$ *on the left matches the action of* $[KgK]^t = [Kg^{-1}K]$ *on the right.*

Proof. For the existence of the duality pairing we refer to [Har66]. We simply prove the formula for the adjoint. We denote by

$$D_{K'}(-) = \underline{\mathrm{RHom}}(-, \mathcal{V}_{-2\rho_{nc},K',\Sigma'''}(-D_{K',\Sigma'''}))$$

the dualizing functor on $S^{tor}_{K',\Sigma'''}$ for any compact open K' (and cone decomposition Σ'''). Let $f = [KgK]$ be a characteristic function to which we associate a Hecke correspondence $S^{tor}_{K,\Sigma} \stackrel{p_1}{\leftarrow} S^{tor}_{K\cap gKg^{-1},\Sigma''} \stackrel{p_2}{\rightarrow} S^{tor}_{K,\Sigma'}$. The action of f on the cohomology arises from a cohomological correspondence (see section 4.2.1):

$$f : p_2^\star \mathcal{V}_{\kappa,K,\Sigma} \stackrel{f_\kappa}{\rightarrow} p_1^\star \mathcal{V}_{\kappa,K,\Sigma} \stackrel{Id\otimes \mathrm{tr}_{p_1}}{\rightarrow} p_1^! \mathcal{V}_{\kappa,K,\Sigma}.$$

We find (since duality switches * and $^!$) that

$$D_{K\cap gKg^{-1}}(f) : p_1^\star D_K(\mathcal{V}_{\kappa,K,\Sigma}) \stackrel{Id\otimes \mathrm{tr}_{p_1}}{\rightarrow} p_1^! D_K(\mathcal{V}_{\kappa,K,\Sigma}) \to p_2^! D_K(\mathcal{V}_{\kappa,K,\Sigma}).$$

Remark that

$$\begin{aligned} p_1^! D_K(\mathcal{V}_{\kappa,K,\Sigma}) &= p_1^\star \mathcal{V}^\vee_{\kappa,K,\Sigma} \otimes p_1^! \mathcal{V}_{-2\rho_{nc},K,\Sigma}(-D_{K,\Sigma}) \\ p_2^! D_K(\mathcal{V}_{\kappa,K,\Sigma'}) &= p_2^\star \mathcal{V}^\vee_{\kappa,K,\Sigma'} \otimes p_2^! \mathcal{V}_{-2\rho_{nc},K,\Sigma}(-D_{K,\Sigma'}) \end{aligned}$$

We have a canonical isomorphism

$$\mathrm{Id} : p_1^! \mathcal{V}_{-2\rho_{nc},\Sigma}(-D_{K,\Sigma}) = p_2^! \mathcal{V}_{-2\rho_{nc},K,\Sigma'}(-D_{K,\Sigma'}),$$

as both sheaves identify with the canonical sheaf of $S^{tor}_{K\cap gKg^{-1},\Sigma''}$. The map $p_1^! D_K(\mathcal{V}_{\kappa,K,\Sigma}) \to p_2^! D_K(\mathcal{V}_{\kappa,K,\Sigma})$ therefore writes $f_\kappa^\vee \otimes \mathrm{Id}$. Let $f^t = [Kg^{-1}K]$. We observe that $f_\kappa^\vee = (f^t)_{-w_{0,M}\kappa}$ by definition (this identity boils down to $A^t = ((A^{-1})^{-1})^t$ for a matrix in GL_n). We now claim that we have a commutative diagram:

$$\begin{array}{ccccc}
p_1^\star D_K(\mathcal{V}_{\kappa,K,\Sigma}) & \xrightarrow{\mathrm{Id}\otimes \mathrm{tr}_{p_1}} & p_1^! D_K(\mathcal{V}_{\kappa,K,\Sigma}) & \xrightarrow{f_\kappa^\vee \otimes \mathrm{Id}} & p_2^! D_K(\mathcal{V}_{\kappa,K,\Sigma}) \\
\uparrow {\scriptstyle \mathrm{Id}} & & \uparrow {\scriptstyle \mathrm{Id}\otimes \mathrm{tr}_{p_1}} & & \uparrow {\scriptstyle \mathrm{Id}\otimes \mathrm{tr}_{p_2}} \\
p_1^\star D_K(\mathcal{V}_{\kappa,K,\Sigma}) & \longrightarrow & p_1^\star D_K(\mathcal{V}_{\kappa,K,\Sigma}) & \xrightarrow{f^t_{-w_{0,M}\kappa-2\rho_{nc}}} & p_2^\star D_K(\mathcal{V}_{\kappa,K,\Sigma})
\end{array}$$

This implies that $D_{K\cap gKg^{-1}}(f) = f^t$. The commutativity of the diagram now boils down to the commutativity of:

$$\begin{array}{ccc}
p_1^! \mathcal{V}_{-2\rho_{nc},K,\Sigma}(-D_{K,\Sigma}) & \xrightarrow{\mathrm{Id}} & p_2^! \mathcal{V}_{-2\rho_{nc},K,\Sigma}(-D_{K,\Sigma}) \\
\uparrow {\scriptstyle \mathrm{tr}_{p_1}} & & \uparrow {\scriptstyle \mathrm{tr}_{p_2}} \\
p_1^\star \mathcal{V}_{-2\rho_{nc},K,\Sigma}(-D_{K,\Sigma}) & \xrightarrow{f^t_{-2\rho_{nc}}} & p_2^\star \mathcal{V}_{-2\rho_{nc},K,\Sigma}(-D_{K,\Sigma})
\end{array}$$

It is sufficient to prove the commutativity outside of the boundary. For any level K, the identification of the canonical sheaf of S_K with $\mathcal{V}_{-2\rho_{nc},K}$ is functorial in the tower of Shimura varieties. Therefore, for the action map $[g] : S_{K\cap gKg^{-1}} \to S_{K\cap g^{-1}Kg}$, we find that the map $[g]^\star \mathcal{V}_{-2\rho_{nc},K\cap g^{-1}Kg} \to \mathcal{V}_{-2\rho_{nc},K\cap gKg^{-1}}$ is the canonical isomorphism between canonical sheaves. We finally deduce that the map $(f^t)_{-2\rho_{nc}} : p_1^\star \mathcal{V}_{-2\rho_{nc},K} \to p_2^\star \mathcal{V}_{-2\rho_{nc},K}$ decomposes as:

$$p_1^\star \mathcal{V}_{-2\rho_{nc},K} \to \mathcal{V}_{-2\rho_{nc},K\cap gKg^{-1}} \to g^\star \mathcal{V}_{-2\rho_{nc},K\cap g^{-1}Kg} \to p_2^\star \mathcal{V}_{-2\rho_{nc},K}$$

where the first map is induced by tr_{p_1}, the second map is the canonical isomorphism, and the last map is induced by $g^\star(\mathrm{tr}_{p_2'})^{-1}$ where $p_2' : S_{K\cap g^{-1}Kg} \to S_K$. This is telling us that the diagram commutes. $\square$

Remark 4.14. One proves more generally that the adjoint of an Hecke operator $[K_1gK_2]$ for two (not necessarily equal) compact open subgroup K_1 and K_2 is $[K_1gK_2]^t = [K_2g^{-1}K_1]$. Details are left to the reader.

4.2.4. *The finite slope part of classical cohomology.* We now assume that $G_{\mathbb{Q}_p}$ is quasi-split. We assume that $K = K^p \times K_p$ where $K_p = K_{p,m,b}$ for $m \geq b \geq 0$, $m > 0$ is one of the subgroups with an Iwahori decomposition introduced in section 3.5.1. We recall that for a choice of $+$ or $-$ we have commutative sub-algebras $\mathcal{H}^{\pm}_{p,m,b}$ of $\mathbb{Z}[K_{p,m,b}\backslash G(\mathbb{Q}_p)/K_{p,m,b}]$. The subalgebra $\mathcal{H}^{\pm}_{p,m,b}$ is generated by the double cosets $[K_{p,m,b}tK_{p,m,b}]$ with $t \in T^{\pm}$. We have isomorphisms $\mathcal{H}^{\pm}_{p,m,b} = \mathbb{Z}[T^{\pm}/T_b]$. We also have the ideals $\mathcal{H}^{\pm\pm}_{p,m,b}$ generated by the double cosets $[K_{p,m,b}tK_{p,m,b}]$ with $t \in T^{\pm\pm}$. The anti-involution of $\mathbb{Z}[K_{p,m,b}\backslash G(\mathbb{Q}_p)/K_{p,m,b}]$ defined by inversion exchanges $\mathcal{H}^{+}_{p,m,b}$ with $\mathcal{H}^{-}_{p,m,b}$ and $\mathcal{H}^{++}_{p,m,b}$ with $\mathcal{H}^{--}_{p,m,b}$.

We let

$$\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})^{\pm,fs} = \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma}) \otimes^L_{\mathbb{Q}[T^{\pm}/T_b]} \mathbb{Q}[T(\mathbb{Q}_p)/T_b]$$

be the finite slope direct factor of $\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})$ for the operators in $[K_{p,m,b}tK_{p,m,b}]$ for $t \in T^{\pm}$. We have a similar definition for cuspidal cohomology. We note that the monoids $T^{\pm}$ have the property that for any $t \in T^{\pm\pm}$ and $s \in T^{\pm}$ there is an $s' \in T^{\pm}$ and an $n > 0$ such that $ss' = t^n$. It follows that the finite slope part can also be described as the finite slope part fo the single operator $[K_{p,m,b}tK_{p,m,b}]$ for any $t \in T^{\pm\pm}$.

We note that the Serre duality pairing of Proposition 4.13 restricts to a duality

$$D_F(\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma},\mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})^{\pm,fs})[-d]\simeq \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma},\mathcal{V}_{-2\rho_{nc}-w_{0,M}\kappa,K^pK_{p,m,b},\Sigma}(-D)^{\mp,fs})$$

on finite slope parts.

Remark 4.15. Assume that the group $G_{\mathbb{Q}_p}$ is a Weil restriction of an unramified group. Then, as explained in remark 3.26 the compact $K_{p,1,0}$ is an Iwahori, and the compact $K_{p,1,1}$ is a pro p Iwahori. Let $K_p=K_{p,1,1}$ or $K_{p,1,0}$. All the Hecke operators $[K_ptK_p]$ for $t\in T^+$ or $t\in T^-$ are already invertible in $\mathbb{Q}[K_p\backslash G(\mathbb{Q}_p)/K_p]$ (see [Vig05, Cor. 1] in the unramified case, and [Vig16, Proposition 4.13] in general) and so

$$\mathrm{R}\Gamma(S^{tor}_{K^pK_p,\Sigma},\mathcal{V}_{\kappa,K^pK_p,\Sigma})^{\pm,fs}=\mathrm{R}\Gamma(S^{tor}_{K^pK_p,\Sigma},\mathcal{V}_{\kappa,K^pK_p,\Sigma})$$

and similarly for cuspidal cohomology.

Next we recall how the finite slope part behaves under certain changes of level. We first recall some classical relations in these Hecke algebras.

Lemma 4.16. *Let $K_1,K_2,K_3\subseteq G(\mathbb{Q}_p)$ be open compact subgroups with Iwahori decompositions $K_i=K_i^-K_i^0K_i^+$. Let $t_1,t_2\in T(\mathbb{Q}_p)$. Suppose that $t_1^{-1}K_1^-t_1\cap t_2K_3^-t_2^{-1}\subseteq K_2^-\subseteq t_1^{-1}K_1^-t_1$, $t_1^{-1}K_1^+t_1\cap t_2K_3^+t_2^{-1}K_2^+\subseteq t_2K_3^+t_3$ and $K_1^0\cap K_3^0\subseteq K_2^0\subseteq K_1^0K_3^0$. Then $[K_1t_1K_2][K_2t_2K_3]=[K_1t_1t_2K_3]$*

Proof. This is an immediate consequence of proposition 4.8 upon noting that our hypotheses imply $K_2=(K_2\cap t_1^{-1}K_1t_1)(K_2\cap t_2K_3t_2^{-1})$ and $t_1^{-1}K_1t_1\cap t_2K_3t_2^{-1}\subseteq K_2$. □

Lemma 4.17. *Let $m'\geq b'\geq 0$ and $m\geq b\geq 0$ satisfy $m'\geq m>0$ and $b'\geq b$.*

(1) *For all $t\in T^+$ we have*

$$[K_{p,m',b'}tK_{p,m',b'}][K_{p,m',b'}1K_{p,m,b}]=[K_{p,m',b'}1K_{p,m,b}][K_{p,m,b}tK_{p,m,b}].$$

(2) *For all $t\in T^-$ we have*

$$[K_{p,m,b}tK_{p,m,b}][K_{p,m,b}1K_{p,m',b'}]=[K_{p,m,b}1K_{p,m',b'}][K_{p,m',b'}tK_{p,m',b'}].$$

(3) *For all $t\in T^{++}$ we have factorizations:*

$$[K_{p,m,b}tK_{p,m,b}]=[K_{p,m,b}tK_{p,m+1,b}][K_{p,m+1,b}1K_{p,m,b}]$$

and

$$[K_{p,m+1,b}tK_{p,m+1,b}]=[K_{p,m+1,b}1K_{p,m,b}][K_{p,m,b}tK_{p,m+1,b}].$$

(4) *For all $t\in T^{--}$ we have factorizations:*

$$[K_{p,m,b}tK_{p,m,b}]=[K_{p,m,b}1K_{p,m+1,b}][K_{p,m+1,b}tK_{p,m,b}]$$

and

$$[K_{p,m+1,b}tK_{p,m+1,b}]=[K_{p,m+1,b}tK_{p,m,b}][K_{p,m,b}1K_{p,m+1,b}].$$

Proof. We note that the second and fourth points are just the transposes of the first and third. The first and third points are immediate consequences of lemma 4.16. □

Here are the consequences on cohomology:

Corollary 4.18. *Let $m'\geq b'\geq 0$ and $m\geq b\geq 0$ satisfy $m'\geq m>0$ and $b'\geq b$.*

(1) *For all $t \in T^{d,+}$, the following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m',b'},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m',b'},\Sigma}) & \xrightarrow{[K_{p,m',b'}tK_{p,m',b'}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m',b'},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m',b'},\Sigma}) \\
\uparrow & & \uparrow \\
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma}) & \xrightarrow{[K_{p,m,b}tK_{p,m,b}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})
\end{array}$$

(2) *For all $m' \geq m$ and $t \in T^{d,-}$, the following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m',b'},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m',b'},\Sigma}) & \xrightarrow{[K_{p,m',b'}tK_{p,m',b'}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m',b'},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m',b'},\Sigma}) \\
\downarrow tr & & \downarrow tr \\
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma}) & \xrightarrow{[K_{p,m,b}tK_{p,m,b}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})
\end{array}$$

(3) *For all m and $t \in T^{d,++}$, there is a factorization:*

$$\begin{array}{ccc}
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m+1,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m+1,b},\Sigma}) & \xrightarrow{[K_{p,m+1,b}tK_{p,m+1,b}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m+1,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m+1,b},\Sigma}) \\
\uparrow & \searrow & \uparrow \\
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma}) & \xrightarrow{[K_{p,m,b}tK_{p,m,b}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})
\end{array}$$

(4) *For all m and $t \in T^{d,--}$, there is a factorization:*

$$\begin{array}{ccc}
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m+1,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m+1,b},\Sigma}) & \xrightarrow{[K_{p,m+1,b}tK_{p,m+1,b}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m+1,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m+1,b},\Sigma}) \\
\downarrow tr & \nearrow & \downarrow tr \\
\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma}) & \xrightarrow{[K_{p,m,b}tK_{p,m,b}]} & \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})
\end{array}$$

(5) *We have the same results for cuspidal cohomology.*

We deduce the following classical corollary

Corollary 4.19. (1) *For all $m' \geq m \geq b$ with $m > 0$, the pullback map*

$$\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})^{+,fs} \to \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m',b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m',b},\Sigma})^{+,fs}$$

and the trace map

$$\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m',b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m',b},\Sigma})^{-,fs} \to \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})^{-,fs}$$

are quasi-isomorphisms, compatible with the action of $\mathbb{Q}[T(\mathbb{Q}_p)/T_b]$, and the same statements are true for cuspidal cohomology. Moreover these isomorphisms are compatible with Serre duality.

(2) *For all $m \geq b' \geq b$ with $m > 0$, the pullback map*

$$\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})^{+,fs} \to (\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b'},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b'},\Sigma})^{+,fs})^{T_b/T_{b'}}$$

and the trace map

$$(\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b'},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b'},\Sigma})^{-,fs})^{T_b/T_{b'}} \to \mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})^{-,fs}$$

are quasi-isomorphisms, compatible with the action of $\mathbb{Q}[T(\mathbb{Q}_p)/T_b]$*, and the same statements are true for cuspidal cohomology. Moreover these isomorphisms are compatible with Serre duality.*

Proof. Immediate from corollary 4.18 and theorem 4.7. □

Now let $\chi : T(\mathbb{Z}_p) \to F^\times$ be a finite order character (enlarging F if necessary.) For all $m \geq b \geq \mathrm{cond}(\chi)$ with $m > 0$, the spaces $\mathrm{R}\Gamma(S^{tor}_{K^pK_{p,m,b},\Sigma}, \mathcal{V}_{\kappa,K^pK_{p,m,b},\Sigma})^{+,fs}[\chi]$ are canonically isomorphic. We denote this space by $\mathrm{R}\Gamma(K^p,\kappa,\chi)^{+,fs}$. We define in the same way $\mathrm{R}\Gamma(K^p,\kappa,\chi)^{-,fs}$, and the cuspidal versions $\mathrm{R}\Gamma(K^p,\kappa,\chi,cusp)^{+,fs}$ and $\mathrm{R}\Gamma(K^p,\kappa,\chi,cusp)^{-,fs}$

These are classical finite slope cohomologies, for a tame level K^p and a M_μ-dominant locally algebraic weight (κ,χ). They satisfy a Serre duality:

$$D_F(\mathrm{R}\Gamma(K^p,\kappa,\chi)^{\pm,fs})[-d] \simeq \mathrm{R}\Gamma(K^p,-2\rho_{nc}-w_{0,M_\mu}\kappa,\chi^{-1},cusp)^{\mp,fs}.$$

4.3. **Jacquet Modules.** In this section we translate the finite slope condition into more representation theoretic terms. We keep assuming that $G_{\mathbb{Q}_p}$ is quasi-split with borel B. We let U be the unipotent radical of B. Let π be a smooth admissible representation of $G(\mathbb{Q}_p)$ with coefficient in a field of characteristic 0. We let $\pi(U) \subseteq \pi$ be the submodule generated by the elements $n.v - v$ for $n \in U(\mathbb{Q}_p)$ and $v \in \pi$. We let $\pi_U = \pi/\pi(U)$ be the Jacquet module of π (with respect to U). This is a smooth admissible representation of $T(\mathbb{Q}_p)$ by [Cas], thm. 3.3.1. Moreover, the functor $\pi \mapsto \pi_U$ is an exact functor by [Cas], prop. 3.3.2. We can define similarly the Jacquet module $\pi_{\overline{U}}$ with respect to $\overline{U}$. Note that conjugation by the longest element w_0 of the Weyl group realizes an isomorphism from π_U to $\pi_{\overline{U}}$. Let $\psi : T(\mathbb{Q}_p) \to \mathbb{C}^\times$ be a continuous character. We let $\iota_B^G(\psi) = \{f : G(\mathbb{Q}_p) \to \mathbb{C},\ \text{smooth},\ f(bg) = \psi(b)f(g)\}$, equipped with the left action induced by right translation of $G(\mathbb{Q}_p)$ on itself. We define similarly $\iota_{\overline{B}}^G(\psi)$.

The adjunction formula of [Cas], thm. 3.2.4 states that

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(\pi, \iota_B^G(\psi)) = \mathrm{Hom}_{T(\mathbb{Q}_p)}(\pi_U, \psi)$$

and

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(\pi, \iota_{\overline{B}}^G(\psi)) = \mathrm{Hom}_{T(\mathbb{Q}_p)}(\pi_{\overline{U}}, \psi).$$

Let $K = K_{p,m,b}$. The algebra $\mathcal{H}^\pm_{p,m,b} = \mathbb{Z}[T^\pm/T_b]$ (by lemma 3.28) acts on $\pi^{K_{p,m,b}}$ and we can define $\pi^{K_{p,m,b},\pm,fs} \subseteq \pi^{K_{p,m,b}}$ as the sub-vector space where the operators $[K_{p,m,b}tK_{p,m,b}]$ for $t \in T^\pm$ act bijectively.

Proposition 4.20. *The natural map* $\pi^{K_{p,m,b},+,fs} \to \pi_U^{T_b}$ *is an isomorphism which is* T^+/T_b *equivariant, and the natural map* $\pi^{K_{p,m,b},-,fs} \to \pi_{\overline{U}}^{T_b}$ *is an isomorphism which is* T^-/T_b *equivariant.*

Proof. See [Cas], Lemma 4.1.1 and proposition 4.1.4. □

Proposition 4.21. *For a smooth irreducible representation* π *of* $G(\mathbb{Q}_p)$*, the following properties are equivalent:*

1. *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},+,fs} \neq 0$*,*
2. *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},-,fs} \neq 0$*,*
3. *There exists a character* ψ *of* $T(\mathbb{Q}_p)$ *such that* $\pi \hookrightarrow \iota_B^G\psi$*,*
4. *There exists a character* ψ' *of* $T(\mathbb{Q}_p)$ *such that* $\pi \hookrightarrow \iota_{\overline{B}}^G\psi'$*.*

Proof. The points (1) and (2) are equivalent because the non-vanishing of $\pi^{K_{p,m,b},\pm,fs}$ is equivalent to the non-vanishing of $\pi_U^{T_b}$ and $\pi_{\overline{U}}^{T_b}$ respectively by proposition 4.20. But conjugation by w_0 realizes an isomorphism between these spaces. Similarly (3) is equivalent to (4). If we assume (3), the adjunction formula shows that $\pi_U \neq 0$, hence there exists b such that $\pi_U^{T_b} \neq 0$ and $\pi^{K_{p,m,b},+,fs} \neq 0$ for any $m \geq b$ by proposition 4.20. Conversely, if $\pi^{K_{p,m,b},+,fs} \neq 0$, then $\pi_U^{T_b} \neq 0$ and by adjunction, there is a non zero map: $\pi \to \iota_B^G \psi$ for a character ψ. Since π is irreducible, this map is injective. □

If one of the equivalent properties of the proposition is satisfied, we say that an irreducible smooth representation π is a *finite slope* representation.

Let π be an admissible representation of $G(\mathbb{Q}_p)$. By adjunction, we have a morphism $\pi \to \iota_B^G \pi_U$ and we let π^{fs} be the image of this morphism. We call π^{fs} the finite slope part of π.

Proposition 4.22. *The following properties are satisfied:*

1. *The $G(\mathbb{Q}_p)$-representations π^{fs} is a direct summand of π.*
2. *Any irreducible factor of π^{fs} is a finite slope representation.*
3. *Any irreducible factor of π which is a finite slope representation lies in π^{fs}.*
4. *π^{fs} is the sub-representation of π generated by the $(\pi)^{K_{p,m,b},\pm,fs}$ for all $m \geq b \geq 0$.*

Proof. By the Bernstein decomposition of the category of smooth representation representation $\pi = \pi' \oplus \pi''$ where π' satisfies properties (1), (2) and (3) of the proposition. We have that $\pi''_U = 0$ because the Jacquet functor is exact and π'' has no finite slope irreducible sub-quotient. We deduce that $\pi_U = \pi'_U$. Moreover the morphism $\pi \to \iota_B^G \pi_U$ factorizes into $\pi' \to \iota_B^G \pi_U$ and it follows again from the exactness of the Jacquet functor that the map $\pi' \to \iota_B^G \pi_U$ is injective. Therefore $\pi' = \pi^{fs}$. Let π''' be the sub-representation of π generated by $(\pi)^{K_{p,m,b},\pm,fs}$ for all $m \geq b \geq 0$. We see that $\pi''' \subseteq \pi^{fs}$. But it follows from proposition 4.20 that $\pi'''_U = \pi^{fs}_U$. Therefore π^{fs}/π''' has trivial Jacquet module, hence contains no finite slope sub-quotient and has to be trivial. □

Let us denote by

$$\mathrm{H}^i(K^p, \kappa) = \mathrm{colim}_{K_p} \mathrm{H}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_{\kappa,K^pK_p,\Sigma}),$$

$$\mathrm{H}^i(K^p, \kappa, cusp) = \mathrm{colim}_{K_p} \mathrm{H}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_{\kappa,K^pK_p,\Sigma}(-D_{K^pK_p,\Sigma}))$$

and

$$\overline{\mathrm{H}}^i(K^p, \kappa) = \mathrm{Im}(\mathrm{H}^i(K^p, \kappa, cusp) \to \mathrm{H}^i(K^p, \kappa)).$$

These are smooth admissible $G(\mathbb{Q}_p)$-representations. We can consider their finite slope parts $\mathrm{H}^i(K^p, \kappa)^{fs}$, $\mathrm{H}^i(K^p, \kappa, cusp)^{fs}$ and $\overline{\mathrm{H}}^i(K^p, \kappa)^{fs}$. These are direct summands of $\mathrm{H}^i(K^p, \kappa)^{fs}$, $\mathrm{H}^i(K^p, \kappa, cusp)^{fs}$ and $\overline{\mathrm{H}}^i(K^p, \kappa)^{fs}$ respectively, and are generated as $G(\mathbb{Q}_p)$-representations by the vector spaces $\mathrm{H}^i(K^p, \kappa, \chi)^{\pm,fs}$, $\mathrm{H}^i(K^p, \kappa, \chi, cusp)^{\pm,fs}$ and $\overline{\mathrm{H}}^i(K^p, \kappa, \chi)^{\pm,fs} = \mathrm{Im}(\mathrm{H}^i(K^p, \kappa, \chi, cusp)^{\pm,fs} \to \mathrm{H}^i(K^p, \kappa, \chi)^{\pm,fs})$ for all characters $\chi : T(\mathbb{Z}_p) \to \overline{F}^\times$.

4.4. **The Hodge-Tate period morphism.** In this section we recall a number of results concerning the Hodge-Tate period morphism and infinite level Shimura varieties. We now assume (unless explicitly mentioned) that F is a finite field extension of $\mathbb{Q}_p$ such that we have an embedding $E \hookrightarrow F$ and such that G splits over F. In this paper, the rationality questions with respect to E are not very important. We will frequently allow ourselves to enlarge F if necessary. Let $\mathcal{S}_K^{an} = (S_K \times \mathrm{Spec}\ F)^{an}$, $\mathcal{S}_K^\star = (S_K^\star \times \mathrm{Spec}\ F)^{an}$, $\mathcal{S}_{K,\Sigma}^{tor} = (S_{K,\Sigma}^{tor} \times \mathrm{Spec}\ F)^{an}$, $\mathcal{FL}_{G,\mu} = (FL_{G,\mu} \times \mathrm{Spec}\ F)^{an}$ (see section 3.3 for the meaning of the superscript an). The first of these spaces is not quasi-compact if the Shimura variety is not proper, the other three spaces are quasi-compact. We will also consider the groups $\mathcal{G}^{an} = (G \times \mathrm{Spec}\ \mathbb{Q}_p)^{an}$, $\mathcal{P}_\mu^{an} = (P_\mu \times \mathrm{Spec}\ F)^{an}$, $\mathcal{M}_\mu^{an} = (M_\mu \times \mathrm{Spec}\ F)^{an}$.

4.4.1. *Inverse limit of adic spaces.* We start by a definition following [SW13], sect. 2.4.

Definition 4.23. *Let $\{\mathcal{X}_i\}_{i \in I}$ be a cofiltered inverse system of locally of finite type adic spaces over $\mathrm{Spa}(F, \mathcal{O}_F)$, with finite transition maps. Let $\mathcal{X}$ be a perfectoid space with compatible maps $\mathcal{X} \to \mathcal{X}_i$.*

We say that $\mathcal{X} \sim \lim_{i \in I} \mathcal{X}_i$ if:

(1) *The maps $\mathcal{X} \to \mathcal{X}_i$ induces an homeomorphism of topological spaces $|\mathcal{X}| = \lim_i |\mathcal{X}_i|$.*
(2) *There is a covering of $\mathcal{X}$ by open affinoids $U = \mathrm{Spa}(A, A^+)$ such that U is the preimage of an affinoid $U_i = \mathrm{Spa}(A_i, A_i^+) \subseteq \mathcal{X}_i$ for a cofinal subset of I and the map $\mathrm{colim}_i A_i \to A$ has dense image.*

If $\mathcal{X} \sim \lim_{i \in I} \mathcal{X}_i$, then the diamond $\lim_i \mathcal{X}_i^\diamondsuit$ is representable by the perfectoid space $\mathcal{X}$ by [SW13], prop. 2.4.5. In particular, $\mathcal{X}$ is unique up to a unique isomorphism.

In the notation of point (2), we see that A^0 is a ring of definition of A (because A is uniform) and A^0 is the completion of $\mathrm{colim}_i A_i^0$ with respect to the p-adic topology.

Definition 4.24. *Let $\{\mathcal{X}_i\}_{i \in I}$ be a cofiltered inverse system of locally of finite type adic spaces over $\mathrm{Spa}(F, \mathcal{O}_F)$ with finite transition maps. Let $\mathcal{X}$ be a perfectoid space and assume that $\mathcal{X} \sim \lim_i \mathcal{X}_i$. We say that an open affinoid subset $U \hookrightarrow \mathcal{X}$ is good if it satisfies the second property of definition 4.23. We say that an open affinoid $U_i \hookrightarrow \mathcal{X}_i$ is pregood if the open subset $\mathcal{X} \times_{\mathcal{X}_i} \mathcal{U}_i$ of $\mathcal{X}$ is good.*

Remark 4.25. We remark that a rational subset of a good open affinoid is also good by [Sch13b], proposition 2.22.

Remark 4.26. It is conjectured in [Sch13b], conjecture 2.24 and proposition 2.26 that any open affinoid in $\mathcal{X}_i$ is pregood.

We end this paragraph with two useful lemmas.

Lemma 4.27. *Let $\{\mathcal{X}_i\}_{i \in I}$ be a cofiltered inverse system of locally of finite type adic spaces over $\mathrm{Spa}(F, \mathcal{O}_F)$ with finite transition maps. For each i let Π_i be the set of connected components of $\mathcal{X}_i$ which we assume to be finite. Let $\Pi = \lim_i \Pi_i$. For any $e \in \Pi$ we get a cofiltered inverse system $\{\mathcal{X}_{i,e}\}$. If there is a perfectoid space $\mathcal{X}$ such that $\mathcal{X} \sim \lim_i \mathcal{X}_i$ then for all $e \in \Pi$, there is a perfectoid space $\mathcal{X}_e \sim \lim_i \mathcal{X}_{i,e}$.*

Proof. We reduce to the affine case. Let $\mathcal{X}_i = \mathrm{Spa}(A_i, A_i^+)$. For each $e \in \Pi_i$, we have $\mathcal{X}_{i,e} = \mathrm{Spa}(A_{i,e}, A_{i,e}^+)$ and $A_i = \prod_{e\in\Pi_i} A_{i,e}$. We may assume that all rings A_i are reduced (by taking the reduction). In particular A_i^0 is open and bounded. This does not affect the $\sim$-limit because perfectoid spaces are reduced. We may also assume that all maps $A_i \to A_j$ are injective for $i, j \in I$ and $j \mapsto i$ (replacing A_i by its image in A_j).

We assume that there is a perfectoid space $\mathcal{X} = \mathrm{Spa}(A, A^+) \sim \lim_i \mathcal{X}_i$. Then A^0 is the p-adic completion of $\widehat{\mathrm{colim}_i A_i^0}^p$ and A^0 is a perfectoid $\mathcal{O}_F$-algebra: there is $\varpi \in A^0$ with $\varpi^p \mid p$ and the Frobenius morphism $\phi : A^0/\varpi^p \to A^0/\varpi^p$ is surjective. Moreover, $A = A^0[1/p]$ and A^+ is the closure of $\mathrm{colim} A_i^+$ in A. By approximation, we may assume that $\varpi \in A_i^0$ and by projection we get an element $\varpi \in A_{i,e}$.

We need to see that the map $\phi : \mathrm{colim}_i A_{i,e}^0/\varpi^p \to \mathrm{colim}_i A_{i,e}^0/\varpi^p$ is surjective. Let $x_{i,e} \in A_{i,e}/\varpi^p$. Since A^0 is perfectoid, we see that there exists $j \mapsto i$ and $y_{j,e} \in A_{j,e}/\varpi^p$ such that $y_{j,e}^p = x_{i,e}$. □

Lemma 4.28. *Let $\{\mathcal{X}_i\}_{i\in I}$ be a cofiltered inverse system of locally of finite type separated adic spaces over a perfectoid field $\mathrm{Spa}(F, \mathcal{O}_F)$ with finite transition maps. Let G be a finite group acting on the inverse system via $\mathrm{Spa}(F, \mathcal{O}_F)$-morphisms. Let $\mathcal{X}$ be a perfectoid space such that $\mathcal{X} = \lim_i \mathcal{X}_i$. Assume that for some index i, we have a G-invariant covering of $\mathcal{X}_i$ by pregood affinoids. Then the categorical quotient $\mathcal{Y}_i = \mathcal{X}_i/G$ is representable by an adic space for a cofinal subset of I, the categorial quotient $\mathcal{Y} = \mathcal{X}/G$ is representable by a perfectoid space, and $\mathcal{Y} \sim \lim_i \mathcal{Y}_i$.*

Proof. We may reduce to the affine case with $\mathcal{X} = \mathrm{Spa}(A, A^+)$ and $\mathcal{X}_i = \mathrm{Spa}(A_i, A_i^+)$. By [Han19], thm 3.5, A^G is perfectoid. It is clear that $\mathrm{colim} A_i^G$ is dense in A^G since we have a projector $A \to A^G$, $a \mapsto \frac{1}{|G|}\sum_{g\in G} g.a$. □

4.4.2. *Siegel Shimura varieties.* We assume in this paragraph that (G, X) is the Siegel Shimura datum $(\mathrm{GSp}_{2g}, \mathcal{H}_g)$. Let $K = K_pK^p \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. The reflex field is $\mathbb{Q}$ and the Shimura variety S_K is a moduli space of abelian varieties A, with a level structure and polarization (prescribed by K).

The Shimura variety S_K carries a right pro-étale $G(\mathbb{Q}_p)$-torsor. Namely, equip $\mathbb{Q}_p^{2g}$ with the standard symplectic form and consider the torsor of isomorphisms $\mathbb{Q}_p^{2g} \to \mathrm{H}_1(A, \mathbb{Q}_p)$, respecting the symplectic forms up to a similitude factor, where A is the universal abelian scheme (defined up to isogeny) and $\mathrm{H}_1(A, \mathbb{Q}_p) = V_p(A)$ is the rational Tate module of A. After choosing a geometric point $\overline{x} \to S_K$, this torsor corresponds to a representation of the algebraic fundamental group $\pi_1(S_K, \overline{x}) \to G(\mathbb{Q}_p)$. The image of this morphism lies in the compact open subgroup $K_p \subset G(\mathbb{Q}_p)$ and the corresponding K_p-torsor is realized geometrically by the tower of Shimura varieties $\lim_{K'_p\subseteq K_p} S_{K'_pK^p}$. By pullback to the adic space $\mathcal{S}_K^{an}$, we get a $\mathcal{G}^{an}(\mathbb{Q}_p)$-torsor $\mathcal{G}_{pet,p}^{an}$. If $K_p \subseteq G(\mathbb{Z}_p)$, this torsor has a $G(\mathbb{Z}_p)$ reduction of group structure that we denote by $\mathcal{G}_{pet,p}$.

The (relative) Hodge-Tate filtration is the exact sequence of pro-étale sheaves over $\mathcal{S}_K^{an}$:

$$0 \to \mathrm{Lie}(A) \otimes_{\mathscr{O}_{\mathcal{S}_K^{an}}} \hat{\mathscr{O}}_{\mathcal{S}_K^{an}} \to \mathrm{H}_1(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{\mathscr{O}}_{\mathcal{S}_K} \to \omega_{A^t} \otimes_{\mathscr{O}_{\mathcal{S}_K^{an}}} \hat{\mathscr{O}}_{\mathcal{S}_K^{an}} \to 0$$

The Hodge-Tate filtration gives a $\mathcal{P}_\mu^{an}$-reduction of structure group $\mathcal{P}_{HT}^{an}$ of the $\mathcal{G}^{an}$-torsor $\mathcal{G}_{pet,p}^{an} \times_{\mathcal{G}^{an}(\mathbb{Q}_p)} \mathcal{G}^{an}$. Namely, we consider trivializations of $\mathrm{H}_1(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{\mathscr{O}}_{\mathcal{S}_K}$ which respect the filtration. This is a right $\mathcal{P}_\mu^{an}$-torsor.

We can consider the pushout $\mathcal{P}^{an}_{HT} \times^{\mathcal{P}^{an}_\mu} \mathcal{M}^{an}_\mu := \mathcal{M}^{an}_{HT}$. This pro-étale torsor actually identifies canonically with (the pull back to the pro-étale site of) of the analytic torsor $\mathcal{M}^{an}_{dR}$ which is the analytification of M_{dR} (see section 4.1.1).

The pro-étale K_p-torsor $\lim_{K'_p} S_{K'_pK_p}$ extends to a pro-Kummer étale K_p-torsor $\lim_{K'_p} S^{tor}_{K'_pK_p,\Sigma}$. We can pull it back to the anaytic space $\mathcal{S}^{tor}_{K,\Sigma}$ (see [DLLZ19] for the definition of the pro-Kummer étale site). By pushout along $K_p \to G(\mathbb{Q}_p)$ we get a pro-Kummer étale $G(\mathbb{Q}_p)$-torsor $\mathcal{G}^{an}_{pet,p}$. If $K_p \subseteq G(\mathbb{Z}_p)$ we also have the pro-Kummer étale $G(\mathbb{Z}_p)$-torsor $\mathcal{G}_{pet,p}$.

Let A_Σ be the semi-abelian scheme over $\mathcal{S}^{tor}_{K,\Sigma}$. The Hodge-Tate exact sequence extends to a sequence over the pro-Kummer étale site

$$0 \to \mathrm{Lie}(A_\Sigma) \otimes_{\mathscr{O}_{\mathcal{S}^{tor}_{K,\Sigma}}} \hat{\mathscr{O}}_{\mathcal{S}^{tor}_{K,\Sigma}} \to \mathrm{H}_1(A_\Sigma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{\mathscr{O}}_{\mathcal{S}^{tor}_{K,\Sigma}} \to \omega_{A^t_\Sigma} \otimes_{\mathscr{O}_{\mathcal{S}^{tor}_{K,\Sigma}}} \hat{\mathscr{O}}_{\mathcal{S}^{an}_K} \to 0$$

Therefore, the torsors $\mathcal{P}^{an}_{HT}$ and $\mathcal{M}^{an}_{HT}$ extend over $\mathcal{S}^{tor}_{K,\Sigma}$. Moreover, again by construction, the torsors $\mathcal{M}^{an}_{HT}$ and $\mathcal{M}^{an}_{dR}$ are canonically identified.

4.4.3. *Perfectoid Siegel Shimura varieties.* By [Sch15], thm. III.3.17 there is a perfectoid space $\mathcal{S}^\star_{K^p} \sim \lim_{K_p} \mathcal{S}^\star_{K^pK_p}$.

By [Sch15], we have $G(\mathbb{Q}_p)$-equivariant map $\pi_{HT} : \mathcal{S}^\star_{K^p} \to \mathcal{FL}_{G,\mu}$. Moreover, there exists an affinoid covering $\mathcal{FL}_{G,\mu} = \cup_i V_i$ such that $\cup_i \pi^{-1}_{HT}(V_i)$ is an affinoid perfectoid covering of $\mathcal{S}^\star_{K^p}$ which satisfies the property (2) of definition 4.23. Note also that for any rational subset $V \subseteq V_i$, we deduce that $\pi^{-1}_{HT}(V)$ is again affinoid perfectoid.

The construction of the map $\pi_{HT} : \mathcal{S}^\star_{K^p} \to \mathcal{FL}_{G,\mu}$ is delicate at the boundary, but over the complement of the boundary $\mathcal{S}^{an}_{K^p} \sim \lim_{K_p} \mathcal{S}^{an}_{K_pK^p}$ it has a simple description which is given below.

By [PS16], for any cone decomposition Σ, there is also a perfectoid space $\mathcal{S}^{tor}_{K^p,\Sigma} \sim \lim_{K_p} \mathcal{S}^{tor}_{K^pK_p,\Sigma}$ (it is important that the cone decomposition does not vary in the limit) and we have a map $\mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{S}^\star_{K^p}$ of perfectoid spaces induced by the map at finite level K_p.

The torsor $\mathcal{G}^{an}_{pet,p}$ becomes trivial over $\mathcal{S}^{tor}_{K^p,\Sigma}$ and we therefore get a Hodge-Tate period map $\pi^{tor}_{HT} : \mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{FL}_{G,\mu}$. Let us explain very concretely how this map is defined. Let $\mathrm{Spa}(R, R^+)$ be a perfectoid affinoid open subset of $\mathcal{S}^{tor}_{K^p,\Sigma}$. We can evaluate the sequence

$$0 \to \mathrm{Lie}(A_\Sigma) \otimes_{\mathscr{O}_{\mathcal{S}^{tor}_{K,\Sigma}}} \hat{\mathscr{O}}_{\mathcal{S}^{tor}_{K,\Sigma}} \to \mathrm{H}_1(A_\Sigma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{\mathscr{O}}_{\mathcal{S}^{tor}_{K,\Sigma}} \to \omega_{A^t_\Sigma} \otimes_{\mathscr{O}_{\mathcal{S}^{tor}_{K,\Sigma}}} \hat{\mathscr{O}}_{\mathcal{S}^{an}_K} \to 0$$

on (R, R^+) (viewed as an object of the pro-Kummer-étale site of $\mathcal{S}^{tor}_{K,\Sigma}$) and use the trivialization $\mathbb{Q}_p^{2g} \simeq \mathrm{H}_1(A_\Sigma, \mathbb{Q}_p)$ to get an exact sequence:

$$0 \to \mathrm{Lie}(A_\Sigma) \otimes R \to R^{2g} \to \omega_{A^t_\Sigma} \otimes R \to 0$$

After localizing, we may even assume that $\mathrm{Lie}(A_\Sigma) \otimes R$ and $\omega_{A^t_\Sigma} \otimes R$ are free R-modules. Now let $0 \to R^g \to R^{2g} \to R^g \to 0$ be the (polarized) chain with automorphism group $P_\mu(R)$. We have that $\mathcal{P}^{an}_{HT}(R, R^+) =$

$$\mathrm{Isom}_{\mathrm{symp}}(0 \to R^g \to R^{2g} \to R^g \to 0, 0 \to \mathrm{Lie}(A_\Sigma) \otimes R \to R^{2g} \to \omega_{A^t_\Sigma} \otimes R \to 0)$$

and $\mathcal{P}^{an}_{HT}(R, R^+) \subseteq \mathrm{Isom}_{\mathrm{symp}}(R^{2g}) = \mathrm{GSp}_{2g}(R)$. This is a right $P_\mu(R)$-torsor and there is an element $x \in G(R)$ such that $\mathcal{P}^{an}_{HT}(R, R^+) = xP_\mu(R)$. The automorphism group of $0 \to \mathrm{Lie}(A_\Sigma) \otimes R \to R^{2g} \to \omega_{A^t_\Sigma} \otimes R \to 0$ is $xP_\mu(R)x^{-1}$. Finally we let $\pi_{HT}^{tor}(\mathrm{Spa}(R,R^+)) = x^{-1} \in FL_{G,\mu}(R)$.

Remark 4.29. We are forced to use x^{-1} above because $FL_{G,\mu} = P_\mu \backslash G$. Note that taking the right quotient by P_μ is natural because right translation on G defines a right G-action of $FL_{G,\mu}$ and the map $\pi_{HT} : \mathcal{S}^\star_{K^p} \to \mathcal{FL}_{G,\mu}$ is equivariant for the right $G(\mathbb{Q}_p)$-action. We chose to define $\mathcal{P}^{an}_{HT}$ as a right $\mathcal{P}^{an}_\mu$-torsor, because we want to identify the torsors $\mathcal{M}^{an}_{HT}$ and $\mathcal{M}^{an}_{dR}$. But in the classical theory, M_{dR} is a right torsor. It means that in our convention, the torsor $\mathcal{P}^{an}_{HT}$ is pulled back via π^{tor}_{HT} from the torsor $\mathcal{G}^{an} \to \mathcal{FL}_{G,\mu}$, $x \mapsto x^{-1}$.

Both maps π^{tor}_{HT} and π_{HT} coincide by construction on the open subset $\mathcal{S}^{an}_{K^p}$. We deduce that π_{HT} factors canonically the map π^{tor}_{HT}. We have a diagram:

$$\begin{array}{ccc} \mathcal{S}^{tor}_{K^p,\Sigma} & & \\ \downarrow & \overset{\pi^{tor}_{HT}}{\searrow} & \\ \mathcal{S}^\star_{K^p} & \xrightarrow{\pi_{HT}} & \mathcal{FL}_{G,\mu} \end{array}$$

The key properties of this diagram that we will use are:

- The pull back of the torsor $\mathcal{G}^{an}/U_{\mathcal{P}^{an}_\mu} \to \mathcal{FL}_{G,\mu}$ via π^{tor}_{HT} is $\mathcal{M}^{an}_{HT}$ and this is canonically identified with the pull back via $\mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{S}^{tor}_{K_pK^p,\Sigma}$ of $\mathcal{M}^{an}_{dR}$.
- The map π_{HT} is affine.

4.4.4. *Formal models of perfectoid Siegel Shimura varieties.* We need to consider formal models of the perfectoid Siegel Shimura varieties in order to be able to use the vanishing result below (theorem 4.40). We first recall a number of statements from [PS16]. Let $K = K_pK^p$. For $K_p = \mathrm{GSp}_{2g}(\mathbb{Z}_p)$, we have natural models over Spec $\mathbb{Z}_p$ for $S_{K_pK^p}$, $S^\star_{K_pK^p}$ and $S^{tor}_{K_pK^p,\Sigma}$ that we denote $\mathbf{S}_{K_pK^p}$, $\mathbf{S}^\star_{K_pK^p}$ and $\mathbf{S}^{tor}_{K_pK^p,\Sigma}$. We can also consider the corresponding p-adic formal schemes $\mathfrak{S}_{K_pK^p}$, $\mathfrak{S}^\star_{K_pK^p}$ and $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$.

Now let $K_p \subseteq \mathrm{GSp}_{2g}(\mathbb{Z}_p)$. We can define $\mathfrak{S}_{K_pK^p}$, $\mathfrak{S}^\star_{K_pK^p}$ and $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$ as the normalizations of $\mathfrak{S}_{\mathrm{GSp}_{2g}(\mathbb{Z}_p)K^p}$, $\mathfrak{S}^\star_{\mathrm{GSp}_{2g}(\mathbb{Z}_p)K^p}$ and $\mathfrak{S}^{tor}_{\mathrm{GSp}_{2g}(\mathbb{Z}_p)K^p,\Sigma}$ in $\mathcal{S}_{K_pK^p}$, $\mathcal{S}^\star_{K_pK^p}$ and $\mathcal{S}^{tor}_{K_pK^p,\Sigma}$ respectively. We denote by A the semi-abelian scheme over $\mathfrak{S}^{tor}_{\mathrm{GSp}_{2g}(\mathbb{Z}_p)K^p,\Sigma}$ (it is defined up to prime-to-p isogeny by our choice of level structure).

Let us denote by $K_{p,n} = \{M \in \mathrm{GSp}_{2g}(\mathbb{Z}_p),\ M = 1 \mod p^n\}$. This is the principal level p^n subgroup. Consider the module $\mathbb{Z}^{2g}$ with canonical basis $e_1, \cdots, e_{2g}$, equipped with the standard symplectic form $\langle , \rangle$ given by $\langle e_i, e_{2g-i+1} \rangle = 1$ and $\langle e_i, e_j \rangle = 0$ if $i + j \neq 2g + 1$. Over $\mathcal{S}_{K_{p,n}K^p}$ we have a symplectic isomorphism $(\mathbb{Z}/p^n\mathbb{Z})^{2g} \to A[p^n]$ (up to a similitude factor), and it extends to a morphism of group schemes over $\mathfrak{S}_{K_{p,n}K^p}$, $\mathbb{Z}/p^n\mathbb{Z}^{2g} \to A[p^n]$. We have the Hodge-Tate morphism $\mathrm{HT}_n : A[p^n] \to \omega_{A^t}/p^n$. Using the prime-to-$p$ polarization, we can identify ω_{A^t} and ω_A. We therefore have sections $\mathrm{HT}_n(e_i) \in \mathrm{H}^0(\mathfrak{S}_{K_{p,n}K^p}, \omega_A/p^n)$. We also have a map $\Lambda^g\mathrm{HT}_n : \Lambda^g(\mathbb{Z}/p^n\mathbb{Z})^{2g} \to \det \omega_A/p^n$. Let $r = \binom{g}{2g}$. Let $f_1, \cdots, f_r$ be a basis of $\Lambda^g\mathbb{Z}^{2g}$ obtained by taking exterior products of $e_1, \cdots, e_{2g}$. We get sections $\Lambda^g\mathrm{HT}_n(f_i) \in \mathrm{H}^0(\mathfrak{S}^\star_{K_{p,n}K^p}, \det \omega_A/p^n)$.

By [PS16], proposition 1.7 and corollary 1.7, the sections $\mathrm{HT}_n(e_i)$ extend to sections $\mathrm{HT}_n(e_i) \in \mathrm{H}^0(\mathfrak{S}^{tor}_{K_{p,n}K^p,\Sigma}, \omega_A/p^n)$ and the sections $\Lambda^g\mathrm{HT}_n(f_i)$ extend to sections $\Lambda^g\mathrm{HT}_n(f_i) \in \mathrm{H}^0(\mathfrak{S}^\star_{K_{p,n}K^p}, \det\omega_A/p^n)$.

It follows that over $\mathfrak{S}^\star_{K_{p,n}K^p}$ we have a morphism $\Lambda^g\mathrm{HT}_n : \Lambda^g(\mathbb{Z}/p^n\mathbb{Z})^{2g} \to \det\omega_A/p^n$. These morphisms satisfy the natural compatibilities as n varies. The cokernel of this morphism is killed by $p^{\frac{g}{p-1}}$ (resp. 4^g if $p=2$) by [Far10], theorem 7. We let $\det\omega_A^{mod,n}$ be the subsheaf of $\det\omega_A$ which is the inverse image of

$$\mathrm{Im}(\Lambda^g\mathrm{HT}_n(\Lambda^g(\mathbb{Z}/p^n\mathbb{Z})^{2g}) \otimes \mathscr{O}_{\mathfrak{S}^\star_{K_{p,n}K^p}} \to \det\omega_A/p^n)$$

in $\det\omega_A$. We have $p^{\frac{g}{p-1}}\det\omega_A \subseteq \det\omega_A^{mod,n} \subseteq \det\omega_A$ (resp. $4^g\det\omega_A \subseteq \det\omega_A^{mod,n} \subseteq \det\omega_A$ if $p=2$) .

Let $f : \mathfrak{S}^\star_{K_{p,n+1}K^p} \to \mathfrak{S}^\star_{K_{p,n}K^p}$ be the projection. We clearly have a map $\det\omega_A^{mod,n+1} \to \mathrm{Im}(f^\star\det\omega_A^{mod,n} \to \det\omega_A)$. This map is an isomorphism if $n \geq \frac{g}{p-1}$ (resp. $n \geq 2g$ if $p=2$). We now assume that n is larger than $n_0 = \frac{g}{p-1}$ (resp. $n_0 = 2g$ if $p=2$). We simply denote $\det\omega_A^{mod,n}$ by $\det\omega_A^{mod}$, this is a subsheaf of $\det\omega_A$. The sheaf $\det\omega_A^{mod}$ is not locally free. We can perform a blow up to make it locally free. Let us start with a definition:

Definition 4.30. *Let $\mathfrak{X}$ be a locally of finite type p-adic formal scheme and let $\mathcal{I}$ be a coherent sheaf of ideals such that $p \in \sqrt{\mathcal{I}}$. We let $\mathrm{BL}_{\mathcal{I}}(\mathfrak{X})$ be the p-adic formal scheme obtained by taking the admissible blow-up of $\mathfrak{X}$ at $\mathcal{I}$. We let $\mathrm{NBL}_{\mathcal{I}}(\mathfrak{X})$ be the normalization of $\mathrm{BL}_{\mathcal{I}}(\mathfrak{X})$, this is the normalized blow-up.*

Remark 4.31. The normalization of a formal scheme is well defined in our context by [Con99], coro. 1.2.3.

Let $\mathcal{I}_n$ be the sheaf of ideals of $\mathscr{O}_{\mathfrak{S}^\star_{K_{p,n}K^p}}$ given by $\mathcal{I}_n = \{a \in \mathscr{O}_{\mathfrak{S}^\star_{K_{p,n}K^p}}, a\det\omega_A \subseteq \det\omega_A^{mod}\}$. We let $\mathfrak{S}^{\star,mod}_{K_{p,n}K^p} = \mathrm{NBL}_{\mathcal{I}_n}(\mathfrak{S}^\star_{K_{p,n}K^p})$. Over $\mathfrak{S}^{\star,mod}_{K_{p,n}K^p}$, the sheaf $\det\omega_A^{mod}$ is locally free and moreover the map

$$\Lambda^g\mathrm{HT}_n : \Lambda^g(\mathbb{Z}/p^n\mathbb{Z})^{2g} \to \det\omega_A/p^n$$

induces a surjective map

$$\Lambda^g\mathrm{HT}'_n : \Lambda^g(\mathbb{Z}/p^n\mathbb{Z})^{2g} \to \det\omega_A^{mod}/p^{n-\frac{g}{p-1}}$$

(resp. $\Lambda^g\mathrm{HT}'_n : \Lambda^g(\mathbb{Z}/p^n\mathbb{Z})^{2g} \to \det\omega_A^{mod}/p^{n-2g}$ if $p=2$).

Lemma 4.32. *Let $n \geq n_0$. We have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{S}^{\star,mod}_{K_{p,n+1}K^p} & \longrightarrow & \mathfrak{S}^\star_{K_{p,n+1}K^p} \\ \downarrow h & & \downarrow f \\ \mathfrak{S}^{\star,mod}_{K_{p,n}K^p} & \longrightarrow & \mathfrak{S}^\star_{K_{p,n}K^p} \end{array}$$

where the vertical maps are finite. Moreover, $h^\star\det\omega_A^{mod} = \det\omega_A^{mod}$ and $h^\star\Lambda^g\mathrm{HT}'_n(f_i) = \Lambda^g\mathrm{HT}'_{n+1}(f_i) \in \mathrm{H}^0(\mathfrak{S}^{\star,mod}_{K_{p,n+1}K^p}, \det\omega_A^{mod}/p^{n-\frac{g}{p-1}})$ (resp. $\in \mathrm{H}^0(\mathfrak{S}^{\star,mod}_{K_{p,n+1}K^p}, \det\omega_A^{mod}/p^{n-2g})$ if $p=2$).

Proof. We first observe that $\mathrm{Im}(f^\star\mathcal{I}_n \to \mathscr{O}_{\mathfrak{S}^\star_{K_{p,n+1}K^p}}) = \mathcal{I}_{n+1}$. It follows that we have maps $\mathrm{BL}_{\mathcal{I}_n}\mathfrak{S}^\star_{K_{p,n}K^p} \times_{\mathfrak{S}^{\star,mod}_{K_{p,n}K^p}} \mathfrak{S}^\star_{K_{p,n+1}K^p} \to \mathrm{BL}_{\mathcal{I}_{n+1}}\mathfrak{S}^\star_{K_{p,n+1}K^p} \to \mathrm{BL}_{\mathcal{I}_n}\mathfrak{S}^\star_{K_{p,n}K^p}$.

The map $\mathrm{BL}_{\mathcal{I}_{n+1}}\mathfrak{S}^\star_{K_{p,n+1}K^p} \to \mathrm{BL}_{\mathcal{I}_n}\mathfrak{S}^\star_{K_{p,n}K^p}$ is therefore finite, and so is the map $h : \mathfrak{S}^{\star,mod}_{K_{p,n+1}K^p} \to \mathfrak{S}^{\star,mod}_{K_{p,n}K^p}$. We have a surjective map $h^\star \det \omega_A^{mod} \to \det \omega_A^{mod}$ of invertible sheaves. This map is therefore an isomorphism. The last compatibility follows from the property that $f^\star \Lambda^g \mathrm{HT}_n(f_i) = \Lambda^g \mathrm{HT}_{n+1}(f_i) \in \mathrm{H}^0(\mathfrak{S}^\star_{K_{p,n+1}K^p}, \det \omega_A/p^n)$. □

We can define similarly $\mathfrak{S}^{\star,mod}_{K_pK^p}$ for $K_p \subseteq K_{p,n_0}$. The ideal $\mathcal{I}_{n_0}$ pulls back to an ideal $\mathcal{I}_{K_p}$ of $\mathfrak{S}^\star_{K_pK^p}$ and we let $\mathfrak{S}^{\star,mod}_{K_pK^p} = \mathrm{NBL}_{\mathcal{I}_{K_p}}(\mathfrak{S}^\star_{K_pK^p})$. If $K_p = K_{p,n}$ for $n \geq n_0$, we recover the previous definition.

We finally let $\mathfrak{S}^{\star,mod}_{K^p} = \lim_n \mathfrak{S}^{\star,mod}_{K_{p,n}K^p}$ where the inverse limit is taken in the category of p-adic formal schemes. This inverse limit exists because the transition morphisms are affine. In the limit we have a map $\Lambda^g\mathrm{HT} : \Lambda^g\mathbb{Z}_p^{2g} \to \det \omega_A^{mod}$ whose linearization is surjective. It follows that we have a morphism $\pi_{HT} : \mathfrak{S}^{\star,mod}_{K^p} \to \mathbb{P}^{r-1}$. Let $X_1, \cdots, X_r$ be the homogeneous coordinates on $\mathbb{P}^{r-1}$. For all $1 \leq i \leq r$, let $\mathfrak{U}_i$ be the formal open subscheme defined by the condition $X_i \neq 0$. Let $\mathcal{U}_i$ be its generic fiber.

Proposition 4.33. (1) *The formal scheme $\mathfrak{S}^{\star,mod}_{K^p}$ is integral perfectoid and its generic fiber is the perfectoid space $\mathcal{S}^\star_{K^p}$.*
(2) *The Hodge-Tate map factors through a map $\pi_{HT} : \mathfrak{S}^{\star,mod}_{K^p} \to \mathfrak{F}\mathfrak{L}_{G,\mu}$ and it induces the Hodge-Tate period map $\pi_{HT} : \mathcal{S}^\star_{K^p} \to \mathcal{FL}_{G,\mu}$ of section 4.4.3 on the generic fiber.*
(3) *For all $1 \leq i \leq r$, we have that $\pi_{HT}^{-1}(\mathcal{U}_i)$ is a good open affinoid subset of $\mathcal{S}^\star_{K^p}$.*

Proof. See [PS16], thm 1.18. This relies on the main theorems of [Sch15]. □

Remark 4.34. For n large enough, the open subsets $\pi_{HT}^{-1}(\mathcal{U}_i)$ come from open subsets $\pi_{HT}^{-1}(\mathcal{U}_i)_{K_{p,n}} \hookrightarrow \mathcal{S}^\star_{K_{p,n}K^p}$. One can define ([Sch15], p. 72) a formal model $\mathfrak{S}^{\star-HT}_{K_{p,n}K^p}$ by gluing the formal schemes Spf $\mathrm{H}^0(\pi_{HT}^{-1}(\mathcal{U}_i)_{K_{p,n}}, \mathscr{O}^+_{\mathcal{S}_{K_{p,n}K^p}})$. We will not use this formal model.

We can perform similar constructions with the toroidal compactification at level $K_p \subseteq K_{p,n_0}$. Namely, the ideal $\mathcal{I}_{K_p}$ pulls back to an ideal $\mathcal{J}_{K_p}$ of $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$. We denote by $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma} = \mathrm{NBL}_{\mathcal{J}_{K_p}}\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$. We have natural morphisms $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma} \to \mathfrak{S}^{\star,mod}_{K_pK^p}$.

Remark 4.35. The space $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma}$ is not exactly the space considered in [PS16]. Namely, in that reference, we considered further blow ups in order to make the sheaf denoted ω_A^{mod} in *loc cit* (the subsheaf of ω_A generated by the image of the Hodge-Tate period map) locally free.

We let $\mathfrak{S}^{tor,mod}_{K^p,\Sigma} = \lim_n \mathfrak{S}^{tor,mod}_{K_{p,n}K^p,\Sigma}$ where the inverse limit is taken in the category of p-adic formal schemes. This inverse limit exists because the transition morphisms are affine.

Proposition 4.36. *The formal scheme $\mathfrak{S}^{tor,mod}_{K^p}$ is integral perfectoid and its generic fiber is $\mathcal{S}^{tor}_{K^p,\Sigma}$.*

Proof. This follows from almost verbatim from [PS16], section A.12, taking into account remark 4.35. □

We now let $\mathcal{U} \hookrightarrow \mathcal{FL}_{G,\mu}$ be a quasi-compact open subset. Our goal is to define formal models for $\pi_{HT}^{-1}(\mathcal{U})$ and $(\pi_{HT}^{tor})^{-1}(\mathcal{U})$.

We first need a formal model for $\mathcal{U}$. By [L90̈], thm. 1.6, there exists an ideal $\mathcal{I}$ of $\mathscr{O}_{\mathfrak{FL}_{G,\mu}}$ and an open subscheme $\mathfrak{U}$ of $\mathrm{NBL}_{\mathcal{I}}(\mathfrak{FL}_{G,\mu})$ such that the generic fiber of $\mathfrak{U}$ is $\mathcal{U}$.

We can define $\mathfrak{S}^{tor,mod}_{K^p,\Sigma,\mathfrak{U}} \to \mathfrak{S}^{\star,mod}_{K^p,\mathfrak{U}}$ which is a formal model for $(\pi_{HT}^{tor})^{-1}(\mathcal{U}) \to \pi_{HT}^{-1}(\mathcal{U})$ as follows. The ideal $\mathcal{I}$ pulls back to ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathfrak{S}^{tor,mod}_{K^p,\Sigma}$ and $\mathfrak{S}^{\star,mod}_{K^p}$ respectively. We now wish to consider the normalized blow up of $\mathfrak{S}^{tor,mod}_{K^p,\Sigma}$ and $\mathfrak{S}^{\star,mod}_{K^p}$ at $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively. We actually show that the ideals come from finite level, perform the normalized blow-up at a finite level, and then pass to the limit.

Lemma 4.37. *The ideal $\mathcal{I}_1$ and $\mathcal{I}_2$ are pull back of ideals $\mathcal{I}_{1,K_p}$ and $\mathcal{I}_{2,K_p}$ of $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma}$ and $\mathfrak{S}^{\star,mod}_{K_{p,n}K^p}$ for K_p small enough.*

Proof. It suffices to prove the claim for $\mathcal{I}_2$. We need to prove that $\mathcal{I}_2$ is generated by $K_{p,n}$-invariant sections for n large enough. First observe that the ideal $\mathcal{I}$ contains p^s for some integer s. Over each standard affine $\mathfrak{U}_i$, we have $\mathcal{I}(\mathfrak{U}_i) = (s_{1,i}, \cdots, s_{k,i})$. We can find sections $s'_{1,i}, \cdots, s'_{k,i} \in \mathrm{H}^0(\pi_{HT}^{-1}(\mathcal{U}_i), \mathscr{O}^+_{\mathcal{S}^\star_{K^p}})$ such that $s'_{j,i} = s_{j,i} \mod p^s$ and $s'_{j,i}$ comes from some finite level $K_{p,n}$ by proposition 4.33, (3). Thus, the $s'_{j,i}$ generate $\mathcal{I}_2$ over the image of $\pi_{HT}^{-1}(\mathcal{U}_i)$ in $\mathcal{S}^\star_{K_{p,n}K^p}$. □

We can therefore consider

$$\mathrm{NBL}_{\mathcal{I}_{1,K_p}}(\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma})$$

and

$$\mathrm{NBL}_{\mathcal{I}_{2,K_p}}(\mathfrak{S}^{\star,mod}_{K_pK^p})$$

for K_p small enough.

We let $\mathrm{NBL}_{\mathcal{I}_1}(\mathfrak{S}^{tor,mod}_{K^p,\Sigma}) = \lim_{K_p} \mathrm{NBL}_{\mathcal{I}_{1,K_p}}(\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma})$ and $\mathrm{NBL}_{\mathcal{I}_1}(\mathfrak{S}^{\star,mod}_{K^p,\Sigma}) = \lim_{K_p} \mathrm{NBL}_{\mathcal{I}_{2,K_p}}(\mathfrak{S}^{\star,mod}_{K_pK^p,\Sigma})$.

We have maps

$$\mathrm{NBL}_{\mathcal{I}_1}(\mathfrak{S}^{tor,mod}_{K^p,\Sigma}) \to \mathrm{NBL}_{\mathcal{I}_2}(\mathfrak{S}^{\star,mod}_{K^p}) \to \mathrm{NBL}_{\mathcal{I}}(\mathfrak{FL}_{G,\mu}).$$

We let $\mathfrak{S}^{tor,mod}_{K^p,\Sigma,\mathfrak{U}}$ and $\mathfrak{S}^{\star,mod}_{K^p,\mathfrak{U}}$ be the preimages of $\mathfrak{U}$. These open formal subschemes come from open formal subschemes $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}}$ and $\mathfrak{S}^{\star,mod}_{K_pK^p,\mathfrak{U}}$ of

$$\mathrm{NBL}_{\mathcal{I}_{1,K_p}}(\mathfrak{S}^{tor,mod}_{K_{p,n}K^p,\Sigma})$$

and

$$\mathrm{NBL}_{\mathcal{I}_{2,K_p}}(\mathfrak{S}^{\star,mod}_{K_{p,n}K^p})$$

for large enough K_p (the equations defining $\mathfrak{U}$ are defined at finite level).

The following theorem shows that the spaces $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$, $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma}$ and $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}}$ admit the usual description in terms of certain formal charts. The case of $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$ is available in the literature. The other cases are deduced from the case of $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$, by tracing down what happens with the various normalized blow ups at the level of formal charts. This is possible because the Hodge-Tate period map behaves nicely for degenerations of abelian varieties, since the Hodge-Tate period morphism for étale and multiplicative p-divisible group is trivial.

Theorem 4.38. (1) *Let $K = K_pK^p$ with $K_p \subseteq \mathrm{GSp}_{2g}(\mathbb{Z}_p)$. We have a decomposition $\mathfrak{S}^{tor}_{K_pK^p,\Sigma} = \coprod_\Phi Z_K(\Phi)$ into locally closed formal subschemes, indexed by certain cusp label representatives Φ.*

(2) *The formal completion $\widehat{\mathfrak{S}^{tor}_{K_pK^p,\Sigma}}^{Z_K(\Phi)}$ admits the following canonical description:*
- *There is a tower of p-adic formal schemes:*
$$\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi) \to \mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$$
where $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi)$ is a torsor under a torus $\mathbf{E}_K(\Phi)$, and $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ is an integral model of a lower dimensional Siegel variety.
In the situation that K is the principal level N congruence subgroup, $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ carries a full level N structure, and $\mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi) \to \mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ is an abelian scheme torsor. It parametrizes semi-abelian schemes $0 \to T \to \mathcal{G} \to A \to 0$ (together with certain level structure) where A is the universal abelian scheme over $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ and T is a split torus of rank $g - \dim A$.
- *There is a twisted torus embedding $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$ depending on Σ.*
- *There is an arithmetic group $\Delta_K(\Phi)$ acting on $X^\star(\mathbf{E}_K(\Phi))$ and on $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \hookrightarrow \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$.*
- *We have a $\Delta_K(\Phi)$-invariant closed subscheme $\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \hookrightarrow \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$.*
- *There is a finite morphism $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}) \to \mathfrak{S}^\star_K$.*
- *There is a series of morphisms:*
$$\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi) \to \mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}).$$
- *We have a canonical isomorphism $Z_K(\Phi) = \Delta_K(\Phi)\backslash \mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$.*
- *We have a canonical isomorphism*
$$\widehat{\mathfrak{S}^{tor}_{K,\Sigma}}^{Z_K(\Phi)} \simeq \Delta_K(\Phi)\backslash\big(\mathbf{S}_{K_\Phi}(\widehat{Q_\Phi, D_\Phi}, \Sigma(\Phi))^{\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))}\big).$$

(3) *For small enough K_p,*
- *the ideal $\mathcal{J}_{K_p}$ restricted to $\widehat{\mathfrak{S}^{tor}_{K,\Sigma}}^{Z_K(\Phi)}$ is the pull back of an ideal $\mathcal{J}_{K_p,\Phi}$ of $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$.*
- *Let $\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) = \mathrm{NBL}_{\mathcal{J}_{K_p,\Phi}}\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$ and $Z^{mod}_K(\phi) = \Delta_K(\phi)\backslash \mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$. We have a decomposition $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma} = \coprod_\Phi Z^{mod}_K(\Phi)$.*
- *Let $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}) = \mathrm{NBL}_{\mathcal{J}_{K_p,\Phi}}(\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}))$. We have a finite morphism $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}) \to \mathfrak{S}^{\star,mod}_{K_pK^p}$.*
- *Let $\mathbf{S}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) = \mathrm{NBL}_{\mathcal{J}_{K_p,\Phi}}\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$. We have a canonical isomorphism*
$$\widehat{\mathfrak{S}^{tor,mod}_{K,\Sigma}}^{Z^{mod}_K(\Phi)} \simeq \Delta_K(\Phi)\backslash\big(\mathbf{S}^{mod}_{K_\Phi}(\widehat{Q_\Phi, D_\Phi}, \Sigma(\Phi))^{\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))}\big).$$

(4) *For all small enough K_p,*

- *The ideal* $\mathcal{I}_{1,K_p}$ *restricted to* $\widehat{\mathfrak{S}^{tor,mod}_{K,\Sigma}}^{Z^{mod}_K(\Phi)}$ *is the pull back of an ideal* $\mathcal{I}_{1,K_p,\Phi}$ *of* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$.
- *We have a decomposition* $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}} = \coprod_\Phi Z^{mod}_K(\Phi)_{\mathfrak{U}}$ *where* $Z^{mod}_K(\Phi)_{\mathfrak{U}}$ *is an open subset of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p}}(Z^{mod}_K(\Phi))$.
- *There is an open subset* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})_{\mathfrak{U}}$ *of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p,\Phi}}(\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ *such that we have a finite morphism* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})_{\mathfrak{U}} \to \mathfrak{S}^{\star,mod}_{K_pK^p,\mathfrak{U}}$.
- *There exists an open subset* $\mathbf{S}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))_{\mathfrak{U}}$ *of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p,\Phi}}(\mathbf{S}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)))$ *and an open subset* $\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))_{\mathfrak{U}}$ *of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p,\Phi}}(\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)))$ *such that we have a canonical isomorphism*

$$\widehat{\mathfrak{S}^{tor,mod}_{K,\Sigma,\mathfrak{U}}}^{Z^{mod}_K(\Phi)_{\mathfrak{U}}} \simeq \Delta_K(\Phi)\backslash\big(\widehat{\mathbf{S}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))_{\mathfrak{U}}}^{\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi,D_\Phi,\Sigma(\Phi))_{\mathfrak{U}}}\big).$$

Remark 4.39. The main observation needed to prove this theorem is the property that the ideals $\mathcal{J}_{K_p}$ and $\mathcal{I}_{1,K_p}$ come from ideals $\mathcal{J}_{K_p,\Phi}$ and $\mathcal{I}_{1,K_p,\Phi}$ on each formal chart (first item in (3) and (4)). This is reminiscent of the concept of well positioned subset or subscheme of [LS18], def. 2.2.1.

Proof. The first two items follow from [Lan17], [MP19] or [PS16] (for principal level structures). The point (3), follows from [PS16], section A.12. It remains to check the point (4). The key property is that the ideal $\mathcal{I}_{1,K_p}$ restricted to $\widehat{\mathfrak{S}^{tor,mod}_{K,\Sigma}}^{Z^{mod}_K(\Phi)}$ is the pull back of an ideal $\mathcal{I}_{1,K_p,\Phi}$ of $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$. The rest follows easily. We argue as follows. Let Φ be a cusp label at some finite level K_pK^p. We can consider the completion of $\mathfrak{S}^{tor,mod}_{K^p,\Sigma}$ at (the inverse image of) $Z^{mod}_{K_pK^p}(\Phi)$. Then we have a map $\widehat{\mathfrak{S}^{tor,mod}_{K^p,\Sigma}}^{Z^{mod}_{K_pK^p}(\Phi)} \to \mathbf{S}^{mod}_{K^p_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ where $\mathbf{S}^{mod}_{K^p_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ is a perfectoid formal scheme attached to a Siegel Shimura datum of lower dimension. Moreover, we have a factorization of the Hodge-Tate period map as follows (compare with [CS19], coro. 4.2.2):

$$\begin{array}{ccc} \widehat{\mathfrak{S}^{tor,mod}_{K^p,\Sigma}}^{Z^{mod}_{K_pK^p}(\Phi)} & \xrightarrow{\pi^{tor}_{HT}} & \mathfrak{FL}_{G,\mu} \\ \downarrow & & \uparrow \\ \mathbf{S}^{mod}_{K^p_\Phi}(G_{\Phi,h}, D_{\Phi,h}) & \xrightarrow{\pi_{HT}} & \mathfrak{FL}_{G_{\Phi,h},\mu_{G_{\Phi,h}}} \end{array}$$

The sheaf of ideals $\mathcal{I}$ of $\mathfrak{FL}$ restricts to a sheaf of ideals $\mathcal{I}_\Phi$ of $\mathfrak{FL}_{G_{\Phi,h},\mu_{G_{\Phi,h}}}$. We deduce that at infinite level, the sheaf $\mathcal{I}_1$ restricted to $\widehat{\mathfrak{S}^{tor,mod}_{K^p,\Sigma}}^{Z^{mod}_{K_pK^p}(\Phi)}$ is the pull back of a sheaf of ideals $\mathcal{I}_{1,\Phi}$ on $\mathbf{S}^{mod}_{K^p_\Phi}(G_{\Phi,h}, D_{\Phi,h})$. We can then prove that $\mathcal{I}_{2,\Phi}$ comes from finite level as in lemma 4.37. □

Theorem 4.40. *Consider the map* $\pi : \mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}} \to \mathfrak{S}^{\star,mod}_{K_pK^p,\mathfrak{U}}$. *Then we have that* $\mathrm{R}^i\pi_\star\mathscr{O}_{\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}}}(-nD) = 0$ *for all* $i > 0$ *and* $n \geq 1$.

Proof. The proof of theorem 4.6 transposes verbatim, given theorem 4.38. □

4.4.5. *The Hodge case.* We fix an embedding $(G, X) \hookrightarrow (\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ of a Hodge type Shimura datum into a Siegel Shimura datum. Let V be a $2g$-dimensional vector space over $\mathbb{Q}$, equipped with a symplectic pairing so that $G \hookrightarrow \mathrm{GSp}_{2g} \hookrightarrow \mathrm{GL}(V)$. We can view $G \hookrightarrow \mathrm{GL}(V)$ as the group stabilizing a finite number of tensors $\{s_\alpha\} \in V^{\otimes}$ (where $V^{\otimes} = \oplus_{m,n \in \mathbb{Z}_{\geq 0}} V^{\otimes m} \otimes (V^\vee)^{\otimes n}$). The group GSp_{2g} will be denoted by $\tilde{G}$ in this section. All the objects corresponding to this group will carry a $\tilde{}$ in general.

Over $S_{K_pK^p}$ we have a pro-étale K_p-torsor represented by the tower of Shimura varieties $\lim_{K'_p} S_{K'_pK^p}$. By pushout along the map $K_p \to G(\mathbb{Q}_p)$ we get a $G(\mathbb{Q}_p)$-torsor.

One can give a more "modular" description of this torsor using the closed embedding $S_K \hookrightarrow \tilde{S}_{\tilde{K}}$, where $\tilde{S}_{\tilde{K}}$ is a Siegel Shimura variety (over E) and the embedding is the one induced by $(G, X) \hookrightarrow (\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ for a suitable compact open subgroup $\tilde{K}$. The tensors $\{s_\alpha\}$ can be used to produce sections $\{s_{\alpha,p} \in \mathrm{H}^0(S_K, \mathrm{H}_1(A, \mathbb{Q}_p)^{\otimes})\}$. More precisely, one first produces tensors $\{s_{\alpha,B} \in \mathrm{H}^0(S_K(\mathbb{C}), \mathrm{H}_1(A, \mathbb{Q})^{\otimes})\}$ using the complex uniformization of $S_K(\mathbb{C})$. They give tensors $\{s_{\alpha,p} \in \mathrm{H}^0(S_K(\mathbb{C}), \mathrm{H}_1(A, \mathbb{Q}_p)^{\otimes})\}$. Lemma 2.3.2 of [CS17] proves that these tensors are defined over E. Therefore, we can consider the torsor of isomorphisms $V \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{H}_1(A, \mathbb{Q}_p)$, preserving all the tensors $s_{\alpha,p}$.

There is also the M_μ-torsor M_{dR} over S_K. We recall its description. The tensors $\{s_\alpha\}$ can be used to produce sections $\{s_{\alpha,dR} \in \mathrm{H}^0(S_K, \mathrm{H}_{1,dR}(A)^{\otimes})\}$. One first define the P_μ^{std}-torsor P_{dR}, to be the the torsor of isomorphisms $V \otimes_{\mathbb{Q}} \mathscr{O}_{S_K} \to \mathrm{H}_{1,dR}(A)$ matching the filtration on V correponding to P_μ^{std} with the Hodge filtration, and preserving the tensors $s_{\alpha,dR}$. By pushout along $P_\mu^{std} \to M_\mu$, we have $M_{dR} = P_{dR} \times^{P_\mu^{std}} M_\mu$.

Remark 4.41. The closed immersion $S_K \hookrightarrow \tilde{S}_{\tilde{K}}$ extends to a closed immersion $M_{dR} \hookrightarrow \tilde{M}_{dR}$ where $\tilde{M}_{dR}$ is the $M_{\tilde{\mu}}$- "de Rham" torsor over $\tilde{S}_{\tilde{K}}$.

By pull back to the analytic space $\mathcal{S}_K^{an}$ we get a $\mathcal{G}^{an}(\mathbb{Q}_p)$-torsor $\mathcal{G}_{pet,p}^{an}$, as well as a $\mathcal{M}_\mu^{an}$-torsor $\mathcal{M}_{dR}^{an}$.

In section 2.3 of [CS17] the authors define two other pro-étale torsors over $\mathcal{S}_K^{an}$: $\mathcal{P}_{HT}^{an}$ and $\mathcal{M}_{HT}^{an}$, under the groups $\mathcal{P}_\mu^{an}$ and respectively $\mathcal{M}_\mu^{an}$. These definitions extend those given in the Siegel case (see section 4.4.2) and use the tensors $s_{\alpha,p}$. Moreover, $\mathcal{P}_{HT}^{an} \times^{\mathcal{P}_\mu^{an}} \mathcal{G}^{an} = \mathcal{G}_{et,p}^{an} \times^{G(\mathbb{Q}_p)} \mathcal{G}^{an}$ and $\mathcal{M}_\mu^{an} = \mathcal{P}_{HT}^{an} \times^{\mathcal{P}_\mu^{an}} \mathcal{M}_\mu^{an}$. By [CS17], prop. 2.3.9, there is a canonical identification of torsors $\mathcal{M}_{dR}^{an}$ and $\mathcal{M}_{HT}^{an}$

4.4.6. *Perfectoid Hodge type Shimura varietes.* By [Sch15], there is a perfectoid space $\mathcal{S}_{K^p}^{an} \sim \lim_{K_p} \mathcal{S}_{K^pK_p}^{an}$. Since the torsor $\mathcal{G}_{pet,p}^{an}$ becomes trivial over $\mathcal{S}_{K^p}^{an}$, we obtain a Hodge-Tate period map ([CS17], thm 2.1.3): $\pi_{HT} : \mathcal{S}_{K^p}^{an} \to \mathcal{FL}_{G,\mu}$ which is $G(\mathbb{Q}_p)$-equivariant.

A key property is that the pull back of the $\mathcal{M}_\mu^{an}$- torsor $\mathcal{G}^{an}/U_{\mathcal{P}_\mu^{an}} \to \mathcal{FL}_{G,\mu}$ via π_{HT} is $\mathcal{M}_{HT}^{an}$ and this is canonically identified with the pull back via $\mathcal{S}_{K^p}^{an} \to \mathcal{S}_{K_pK^p}^{an}$ of $\mathcal{M}_{dR}^{an}$.

Moreover, the relation with the Siegel datum is expressed by the following diagram, where the horizontal maps are closed immersions:

$$\begin{array}{ccc} \mathcal{S}^{an}_{K^p} & \longrightarrow & \tilde{\mathcal{S}}^{an}_{\tilde{K}^p} \\ \downarrow & & \downarrow \\ \mathcal{FL}_{G,\mu} & \longrightarrow & \mathcal{FL}_{\tilde{G},\tilde{\mu}} \end{array}$$

4.4.7. *Compactifications in the Hodge case.* For any compact open $K \subset G(\mathbb{A}_f)$, we have the minimal compactification $\mathcal{S}^\star_K$ and there is a finite surjective map $\mathcal{S}^\star_K \to \mathcal{S}^\star_{\overline{K}}$ where $\mathcal{S}^\star_{\overline{K}}$ is defined before [Sch15], thm. IV. 1.1. This is the schematic image of the morphism $\mathcal{S}^\star_K \to \tilde{\mathcal{S}}^\star_{\tilde{K}}$ where $\tilde{\mathcal{S}}^\star_{\tilde{K}}$ is the minimal compactification of the Shimura variety for $\tilde{G}$ and $\tilde{K}$ is a small enough compact open subgroup of $\tilde{G}(\mathbb{A}_f)$ such that $\tilde{K} \cap G(\mathbb{A}_f) = K$. By [Sch15], thm. IV. 1.1 there is a perfectoid space $\mathcal{S}^\star_{\overline{K^p}} \sim \lim_{K_p} \mathcal{S}^\star_{\overline{K_pK^p}}$ and there is a map $\pi_{HT} : \mathcal{S}^\star_{\overline{K^p}} \to \mathcal{FL}_{G,\mu}$. Strictly speaking the map constructed in [Sch15], took values in $\mathcal{FL}_{\tilde{G},\tilde{\mu}}$. Since $\mathcal{FL}_{G,\mu} \hookrightarrow \mathcal{FL}_{\tilde{G},\tilde{\mu}}$ is Zariski closed and the map is known to factors through $\mathcal{FL}_{G,\mu}$ over the Zariski dense open subset $\mathcal{S}^{an}_{K^p}$ by [CS17], thm 2.1.3, we deduce that we have a map $\pi_{HT} : \mathcal{S}^\star_{\overline{K^p}} \to \mathcal{FL}_{G,\mu}$.

The tower $\{\mathcal{S}^\star_{\overline{K_pK^p}}\}_{K_p}$ carries a $G(\mathbb{Q}_p)$-action, the space $\mathcal{S}^\star_{\overline{K^p}}$ therefore inherits a $G(\mathbb{Q}_p)$-action and the map π_{HT} is $G(\mathbb{Q}_p)$-equivariant. Moreover, the map π_{HT} is affine in the sense that there exists an affinoid covering $\mathcal{FL}_{G,\mu} = \cup_i V_i$ such each $\pi_{HT}^{-1}(V_i)$ is a good affinoid perfectoid open subset of $\mathcal{S}^\star_{K^p}$ (see definition 4.24).

We also know that there is a perfectoid space $\mathcal{S}^\star_{K^p} = \lim_{K_p} \mathcal{S}^{\star,\diamondsuit}_{K^pK_p}$ (this last inverse limit is taken in the category of diamonds) by [BS19], thm. 1.16. We therefore have a $G(\mathbb{Q}_p)$-equivariant map $\mathcal{S}^\star_{K^p} \to \mathcal{S}^\star_{\overline{K^p}}$. It is not known whether $\mathcal{S}^\star_{K^p} \sim \lim \mathcal{S}^\star_{K^pK_p}$ in general.

By [Lan20], for a cofinal subset of cone decompositions Σ, we have a perfectoid space $\mathcal{S}^{tor}_{K^p,\Sigma} \sim \lim \mathcal{S}^{tor}_{K^pK_p,\Sigma}$. More precisely, for each such cone decomposition Σ, there exists a cone decomposition $\tilde{\Sigma}$ and a closed immersion of perfectoid spaces $\mathcal{S}^{tor}_{K^p,\Sigma} \hookrightarrow \mathcal{S}^{tor}_{\tilde{K}^p,\tilde{\Sigma}}$. Let us call call these cone decompositions *perfect* cone decompositions because they give rise to perfectoid toroidal compactifications.

Remark 4.42. We did not prove that there is a perfectoid space $\mathcal{S}^{tor}_{K^p,\Sigma} \sim \lim \mathcal{S}^{tor}_{K^pK_p,\Sigma}$ for any cone decomposition Σ. It seems likely that one could reproduce the argument of [PS16] in the Hodge case, using the explicit description of the boundary of the integral toroidal compactifications given in [MP19].

For a perfect cone decomposition Σ, we have a series of maps $\mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{S}^\star_{K^p} \to \mathcal{S}^\star_{\overline{K^p}}$ and we therefore get a map $\pi^{tor}_{HT} : \mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{FL}_{G,\mu}$.

The relation to the Siegel Shimura varieties is given by the following diagram where all horizontal maps are closed immersions:

$$\begin{array}{ccc}
\mathcal{S}^{tor}_{K^p,\Sigma} & \longrightarrow & \tilde{\mathcal{S}}^{tor}_{\tilde{K}^p,\tilde{\Sigma}} \\
\downarrow & & \\
\mathcal{S}^{\star}_{K^p} & & \downarrow \\
\downarrow & & \\
\mathcal{S}^{\star}_{\overline{K^p}} & \longrightarrow & \tilde{\mathcal{S}}^{\star}_{\tilde{K}^p} \\
\downarrow & & \downarrow \\
\mathcal{FL}_{G,\mu} & \longrightarrow & \mathcal{FL}_{\tilde{G},\tilde{\mu}}
\end{array}$$

Using the map π^{tor}_{HT} we can define a canonical extension of the torsors $\mathcal{P}^{an}_{HT}$ and $\mathcal{M}^{an}_{HT}$ to $\mathcal{S}^{tor}_{K^p,\Sigma}$, by simply by pulling back the universal $\mathcal{P}^{an}_{\mu}$-torsor over $\mathcal{FL}_{G,\mu}$ and pushing out along the map $\mathcal{P}^{an}_{\mu} \to \mathcal{M}^{an}_{\mu}$. There is also the torsor $\mathcal{M}^{an}_{dR}$ over $\mathcal{S}^{tor}_{K,\Sigma}$ which is pulled back from the map $\mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{S}^{tor}_{K_pK^p,\Sigma}$. The following proposition is corollary 5.2 of [EH19].

Proposition 4.43 ([EH19])**.** *The torsors $\mathcal{M}^{an}_{HT}$ and $\mathcal{M}^{an}_{dR} \times_{\mathcal{S}^{tor}_{K,\Sigma}} \mathcal{S}^{tor}_{K^p,\Sigma}$ are canonically isomorphic.*

Proof. Let us denote by $\tilde{M}$ the restriction to $\mathcal{S}^{tor}_{K^p,\Sigma}$ of the torsor $\tilde{\mathcal{M}}^{an}_{dR} = \tilde{\mathcal{M}}^{an}_{HT}$ defined over the Siegel perfectoid Shimura variety $\tilde{\mathcal{S}}_{\tilde{K}^p,\tilde{\Sigma}}$. By construction, $\tilde{M}$ is a $\mathcal{M}^{an}_{\tilde{\mu}}$-torsor. The two torsors $\mathcal{M}^{an}_{HT}$ and $\mathcal{M}^{an}_{dR}$ are $\mathcal{M}^{an}_{\mu}$-reductions of this torsor. They coincide over $\mathcal{S}^{an}_{K^p}$ by [CS17], prop. 2.3.9. But $\mathcal{M}^{an}_{HT}$ (resp. $\mathcal{M}^{an}_{dR}$) is equal to the Zariski closure of $\mathcal{M}^{an}_{HT}|_{\mathcal{S}^{an}_{K^p}}$ (resp. $\mathcal{M}^{an}_{dR}|_{\mathcal{S}^{an}_{K^p}}$) in $\tilde{M}$. Therefore, these torsors have to coincide everywhere. □

Remark 4.44. The isomorphism of torsors is Hecke equivariant.

4.4.8. *Integral models in the Hodge case.* We need to consider formal models. This section is entirely parallel to section 4.4.4. Let $K = K_pK^p$ with $K = \tilde{K} \cap G(\mathbb{A}_f)$. Suppose that $\tilde{K}_p \subseteq \tilde{G}(\mathbb{Z}_p)$. We can define $\mathfrak{S}_{K_pK^p}$, $\mathfrak{S}^{\star}_{K_pK^p}$ and $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$ as the normalizations of $\tilde{\mathfrak{S}}_{\tilde{K}}$, $\tilde{\mathfrak{S}}^{\star}_{\tilde{K}}$ and $\tilde{\mathfrak{S}}^{tor}_{\tilde{K},\tilde{\Sigma}}$ in $\mathcal{S}_{K_pK^p}$, $\mathcal{S}^{\star}_{K_pK^p}$ and $\mathcal{S}^{tor}_{K_pK^p,\Sigma}$ respectively.

Similarly, for K_p small enough, we define $\mathfrak{S}^{\star,mod}_{K_pK^p}$, $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma}$ as the normalizations of $\tilde{\mathfrak{S}}^{\star,mod}_{\tilde{K}}$ and $\tilde{\mathfrak{S}}^{tor,mod}_{\tilde{K},\Sigma}$ in $\mathcal{S}^{\star}_{K_pK^p}$ and $\mathcal{S}^{tor}_{K_pK^p,\Sigma}$ respectively. Alternatively, these can also be constructed as normalized blow ups of $\mathfrak{S}^{\star}_{K_pK^p}$ and $\mathfrak{S}^{tor}_{K_pK^p,\Sigma}$ for the ideals $\mathcal{I}_{K_p}$ and $\mathcal{J}_{K_p}$ which are the pull back of the ideals $\mathcal{I}_{\tilde{K}_p}$ and $\mathcal{J}_{\tilde{K}_p}$ from $\tilde{\mathfrak{S}}^{\star}_{\tilde{K}}$ and $\tilde{\mathfrak{S}}^{tor}_{\tilde{K},\Sigma}$.

We let $\mathfrak{S}^{\star,mod}_{K^p} = \lim_{K_p} \mathfrak{S}^{\star,mod}_{K_pK^p}$ where the inverse limit is taken in the category of p-adic formal schemes. This inverse limit exists because the transition morphisms are affine. In the limit we have a map $\mathfrak{S}^{\star,mod}_{K^p} \to \tilde{\mathfrak{S}}^{\star,mod}_{\tilde{K}^p}$ and therefore a map $\pi_{HT} : \mathfrak{S}^{\star,mod}_{K^p} \to \mathfrak{FL}_{\tilde{G},\mu_{\tilde{G}}}$.

Proposition 4.45. *The Hodge-Tate map factors through a map $\pi_{HT} : \mathfrak{S}^{\star,mod}_{K^p} \to \mathfrak{FL}_{G,\mu}$.*

Proof. One first checks that the space $\mathfrak{S}^{mod}_{K^p}$ (the complement of the boundary in $\mathfrak{S}^{\star,mod}_{K^p}$) is integral perfectoid and its generic fiber is the quasi-compact open perfectoid Shimura variety $\mathcal{S}_{K^p}$. This follows from the Siegel case, using that $\mathcal{S}_{K^p} \hookrightarrow \tilde{S}_{\tilde{K}^p}$ is a Zariski closed immersion. Therefore the factorization of the period morphism through $\mathfrak{F}\mathfrak{L}_{G,\mu}$ holds over the Zariski dense subspace $\mathfrak{S}^{mod}_{K^p}$, and thus everywhere. □

Remark 4.46. We do not know if $\mathfrak{S}^{\star,mod}_{K^p}$ is integral perfectoid.

We let $\mathfrak{S}^{tor,mod}_{K^p,\Sigma} = \lim_{K_p} \mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma}$ where the inverse limit is taken in the category of p-adic formal schemes. This inverse limit exists because the transition morphisms are affine.

Proposition 4.47. *Assume that* Σ *is perfect. The formal scheme* $\mathfrak{S}^{tor,mod}_{K^p,\Sigma}$ *is integral perfectoid and its generic fiber is* $\mathcal{S}^{tor}_{K^p,\Sigma}$.

Proof. This follows from the Siegel case. □

Remark 4.48. We do not know if $\mathfrak{S}^{tor,mod}_{K^p,\Sigma}$ is integral perfectoid for all Σ.

We now let $\mathcal{U} \hookrightarrow \mathcal{FL}_{G,\mu}$ be a quasi-compact open subset. It is induced by an open subset $\tilde{\mathcal{U}}$ of $\mathcal{FL}_{\tilde{G},\tilde{\mu}}$. Our goal is to define formal models for $\pi_{HT}^{-1}(\mathcal{U})$ and $(\pi_{HT}^{tor})^{-1}(\mathcal{U})$.

There exists an ideal $\tilde{\mathcal{I}}$ of $\mathscr{O}_{\mathfrak{F}\mathfrak{L}_{\tilde{G},\mu_{\tilde{G}}}}$ and an open subscheme $\tilde{\mathfrak{U}}$ of $\mathrm{NBL}_{\tilde{\mathcal{I}}}(\mathfrak{F}\mathfrak{L}_{\tilde{G},\mu_{\tilde{G}}})$ such that the generic fiber of $\tilde{\mathfrak{U}}$ is $\tilde{\mathcal{U}}$.

For K_p small enough, we define $\mathfrak{S}^{\star,mod}_{K_pK^p,\mathfrak{U}}$, $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}}$ as the normalizations of $\tilde{\mathfrak{S}}^{\star,mod}_{\tilde{K},\tilde{\mathfrak{U}}}$ and $\tilde{\mathfrak{S}}^{tor,mod}_{\tilde{K},\tilde{\Sigma},\tilde{\mathfrak{U}}}$ in $\pi_{HT}^{-1}(\mathcal{U})$ and $(\pi_{HT}^{tor})^{-1}(\mathcal{U})$ respectively. Alternatively, these can also be constructed as suitable opens in normalized blow ups of $\mathfrak{S}^{\star,mod}_{K_pK^p}$ and $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma}$ at the ideals $\mathcal{I}_{1,K_p}$ and $\mathcal{I}_{2,K_p}$ which are the pull back of the ideals $\mathcal{I}_{1,\tilde{K}_p}$ and $\mathcal{I}_{2,\tilde{K}_p}$ from $\tilde{\mathfrak{S}}^{\star,mod}_{\tilde{K}}$ and $\tilde{\mathfrak{S}}^{tor,mod}_{\tilde{K},\tilde{\Sigma}}$.

Theorem 4.49.
(1) *Let* $K = K_pK^p$ *with* $K_p \subseteq \tilde{G}(\mathbb{Z}_p)$. *We have a decomposition* $\mathfrak{S}^{tor}_{K_pK^p,\Sigma} = \coprod_\Phi Z_K(\Phi)$ *into locally closed formal subschemes, indexed by certain cusp label representatives* Φ.
(2) *The formal completion* $\widehat{\mathfrak{S}^{tor}_{K_pK^p,\Sigma}}^{Z_K(\Phi)}$ *admits the following canonical description:*
   - *There is a tower of p-adic formal schemes:*
   $$\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi) \to \mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$$
   where $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi)$ *is a torsor under a torus* $\mathbf{E}_K(\Phi)$, *and* $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ *is an integral model of a lower dimensional Shimura variety.*
   - *There is a twisted torus embedding* $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \to \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$ *depending on* Σ.
   - *There is an arithmetic group* $\Delta_K(\Phi)$ *acting on* $X^\star(\mathbf{E}_K(\Phi))$ *and on* $\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi) \hookrightarrow \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$.
   - *We have a* $\Delta_K(\Phi)$*-invariant closed subscheme* $\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \hookrightarrow \mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$.
   - *There is a finite morphism* $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}) \to \mathfrak{S}^\star_K$.

- *There is a series of morphisms:*

$$\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) \to \mathbf{S}_{K_\Phi}(\overline{Q}_\Phi, \overline{D}_\Phi) \to \mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}).$$

- *We have a canonical isomorphism* $Z_K(\Phi) = \Delta_K(\Phi)\backslash \mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$.
- *We have a canonical isomorphism*

$$\widehat{\mathfrak{S}^{tor}_{K,\Sigma}}^{Z_K(\Phi)} \simeq \Delta_K(\Phi)\backslash\big(\mathbf{S}_{K_\Phi}(\widehat{Q_\Phi, D_\Phi}, \Sigma(\Phi))^{\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))}\big).$$

(3) *For small enough* K_p,

- *the ideal* $\mathcal{J}_{K_p}$ *restricted to* $\widehat{\mathfrak{S}^{tor}_{K,\Sigma}}^{Z_K(\Phi)}$ *is the pull back of an ideal* $\mathcal{J}_{K_p,\Phi}$ *of* $\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$.
- *Let* $\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) = \mathrm{NBL}_{\mathcal{J}_{K_p,\Phi}}\mathbf{Z}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$ *and* $Z^{mod}_K(\phi) = \Delta_K(\phi)\backslash \mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$. *We have a decomposition* $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma} = \coprod_\Phi Z^{mod}_K(\Phi)$.
- *Let* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}) = \mathrm{NBL}_{\mathcal{J}_{K_p,\Phi}}(\mathbf{S}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}))$. *We have a finite morphism* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h}) \to \mathfrak{S}^{\star,mod}_{K_pK^p}$.
- *Let* $\mathbf{S}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)) = \mathrm{NBL}_{\mathcal{J}_{K_p,\Phi}}\mathbf{S}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))$. *We have a canonical isomorphism*

$$\widehat{\mathfrak{S}^{tor,mod}_{K,\Sigma}}^{Z^{mod}_K(\Phi)} \simeq \Delta_K(\Phi)\backslash\big(\mathbf{S}^{mod}_{K_\Phi}(\widehat{Q_\Phi, D_\Phi}, \Sigma(\Phi))^{\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))}\big).$$

(4) *For all small enough* K_p,

- *The ideal* $\mathcal{I}_{1,K_p}$ *restricted to* $\widehat{\mathfrak{S}^{tor,mod}_{K,\Sigma}}^{Z^{mod}_K(\Phi)}$ *is the pull back of an ideal* $\mathcal{I}_{1,K_p,\Phi}$ *of* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$.
- *We have a decomposition* $\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}} = \coprod_\Phi Z^{mod}_K(\Phi)_{\mathfrak{U}}$ *where* $Z^{mod}_K(\Phi)_{\mathfrak{U}}$ *is an open subset of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p}}(Z^{mod}_K(\Phi))$.
- *There is an open subset* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})_{\mathfrak{U}}$ *of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p,\Phi}}(\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})$ *such that we have a finite morphism* $\mathbf{S}^{mod}_{K_\Phi}(G_{\Phi,h}, D_{\Phi,h})_{\mathfrak{U}} \to \mathfrak{S}^{\star,mod}_{K_pK^p,\mathfrak{U}}$.
- *There exists an open subset* $\mathbf{S}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))_{\mathfrak{U}}$ *of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p,\Phi}}(\mathbf{S}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)))$ *and an open subset* $\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))_{\mathfrak{U}}$ *of* $\mathrm{NBL}_{\mathcal{I}_{1,K_p,\Phi}}(\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi)))$ *such that we have a canonical isomorphism*

$$\widehat{\mathfrak{S}^{tor,mod}_{K,\Sigma,\mathfrak{U}}}^{Z^{mod}_K(\Phi)_{\mathfrak{U}}} \simeq \Delta_K(\Phi)\backslash\big(\mathbf{S}^{mod}_{K_\Phi}(\widehat{Q_\Phi, D_\Phi}, \Sigma(\Phi))_{\mathfrak{U}}^{\mathbf{Z}^{mod}_{K_\Phi}(Q_\Phi, D_\Phi, \Sigma(\Phi))_{\mathfrak{U}}}\big).$$

Proof. The first two items follow from [MP19]. The point (3) and the point (4) follow from the analogous statement in theorem 4.38. □

Theorem 4.50. *Consider the map* $\pi : \mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}} \to \mathfrak{S}^{\star,mod}_{K_pK^p,\mathfrak{U}}$. *Then we have that* $\mathrm{R}^i\pi_\star\mathscr{O}_{\mathfrak{S}^{tor,mod}_{K_pK^p,\Sigma,\mathfrak{U}}}(-nD) = 0$ *for all* $i > 0$ *and* $n \geq 1$.

Proof. The proof of theorem 4.6 transposes verbatim, given theorem 4.49. □

4.4.9. *Abelian type Shimura varieties.* We now extend part of the picture to abelian type Shimura varieties (G, X) under the mild assumption that $Z(G)^0$ is split by a CM field.

We first deal with the case of tori. So let (T, X) be a Shimura datum with T a torus split by a CM field. There is a perfectoid space $\mathcal{S}_{K^p} \sim \lim_{K_p} \mathcal{S}_{K_pK^p}$. This perfectoid space is actually a finite union of adic spectra of perfectoid fields. The flag variety $\mathcal{FL}_{T,\mu}$ is a point, and the Hodge-Tate period map is the projection to this point. The T^c-torsor T_{dR} over $S_{K_pK^p}$ pulls back to a $\mathcal{T}^{c,an}$-tosor $\mathcal{T}_{dR}$ on the adic space $\mathcal{S}_{K_pK^p}$.

Lemma 4.51. *The pull back of the torsor $\mathcal{T}_{dR}$ to $\mathcal{S}_{K^p}$ is canonically trivial.*

Proof. We reduce to the case that $T = T^c$. Then the Shimura variety parametrizes a CM motive M and this becomes a consequence of the comparison theorem de Rham-étale for CM motives. See [Lov17], prop. 3.1.8. □

Remark 4.52. Of course the trivialization is Hecke-equivariant.

We now consider an abelian Shimura datum (G, X). This means that there is a Hodge type Shimura datum (G_1, X_1) such that the associated connected Shimura datum (G^{ad}, X^+) and (G_1^{ad}, X_1^+) are isomorphic and we have a central isogeny $G_1^{der} \to G$.

There is actually a nice way to connect the datum (G, X) and (G_1, X_1) according to [Lov17], section 4.6 . Let us fix a field E which contains the reflex fields of both Shimura datum. One can construct a diagram of Shimura data

$$\begin{array}{ccc} (B_1, X_{B_1}) & \longrightarrow & (B, X_B) \\ \downarrow & & \downarrow \\ (G_1, X_1) & & (G, X) \end{array}$$

where $B_1 \to G_1$ and $B \to G$ are epimorphisms, inducing isomorphisms $B_1^{der} \simeq G_1^{der}$, $B^{der} \simeq G^{der}$ and the map $B_1 \to B$ is a central isogeny.

Actually $B_1 = G_1 \times_{G_1^{ab}} \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ for a suitable map $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \to G_1^{ab}$ induced by the co-character μ_{G_1}. We let $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$. Moreover, the connected Shimura datum $(B_1^{der}, X_{B_1}^+)$ and (G_1^{der}, X_1^+) are isomorphic, as well as the connected Shimura datum (B^{der}, X_B^+) and (G^{der}, X^+).

Since we are going to deal with several Shimura varieties in this paragraph, we change a bit our notation and the Shimura variety attached to a datum (G, X) and compact open K is $S(G, X)_K$ rather than S_K, we also write $M_{dR}(G, X)$ rather than M_{dR}, etc.

Let us first state a couple of classical results. We choose connected components of $X_{B_1}^+$, X_B^+, X_1^+ and X^+ compatibly with our morphisms of Shimura datum. This allows us to identify the set of geometric connected component of a Shimura variety $S(H, X_H)_K$ with $\overline{H(\mathbb{Q})_+} \backslash H(\mathbb{A}_f)/K$ (for $H \in \{G, G_1, B, B_1\}$). We let $S^0(H, X_H)_K$ be the connected component corresponding to the class of 1 (we allow ourselves to extend the base field). We adopt similar definitions for the minimal and toroidal compactification.

Theorem 4.53. *The following points are satisfied:*

1. *Let $f : B_1 \to G_1$. The towers of connected component of the Shimura variety $\lim_K S^0(B_1, X_{B_1})_{f^{-1}K}$ and $\lim_K S^0(G_1, X_1)_K$ are canonically isomorphic. The same holds for the minimal compactifications and the toroidal compactifications.*
2. *Let K be a compact open subgroup of $B_1(\mathbb{A}_f)$ and Σ be a cone decomposition. We have morphisms $\pi_1 : S^{tor}(B_1, X_{B_1})_{K,\Sigma} \to S^{tor}(G_1, X_1)_{K_1,\Sigma_1}$*

and $\pi_2 : S^{tor}(B_1, X_{B_1})_{K,\Sigma} \to S(T, X_T)_{K_2}$ (for suitable compact open subgroups K_1 and K_2 and cone decomposition Σ_1). The torsor $M_{dR}(B_1, X_{B_1})$ over $S^{tor}(B_1, X_{B_1})_{K_pK^p,\Sigma}$ is the contracted product $\pi_1^\star M_{dR}(G_1, X_1) \times_{M_\mu^{ab,c}} \pi_2^\star M_{dR}(T, X_T)$.

(3) *Let $g : B_1 \to G$. Let K be a compact open subgroup of $G_1(\mathbb{A}_f)$. The geometric connected component of the Shimura variety $S^0(G, X)_K$ is a quotient of a geometric connected component of a Shimura variety $S^0(B_1, X_{B_1})_{g^{-1}(K)}$ by a finite group $\Delta(K)$. The same holds true for the minimal and toroidal compactification.*

(4) *Let Σ be a cone decomposition. The torsor $M_{dR}(G, X)$ over $S^{tor,0}(G, X)_{K,\Sigma}$ is the quotient by $\Delta(K)$ of the torsor*

$$M_{dR}(B_1, X_1) \times^{M_{\mu^c_{B_1}}} M^c_{\mu_G}$$

over $S^{tor,0}(B_1, X_{B_1})_{g^{-1}K,\Sigma}$.

Proof. For the first point, we have a map $S^0(B_1, X_{B_1})_{f^{-1}K} \to S^0(G_1, X_1)_K$ and we can check that it becomes an isomorphism after taking the limit over K by reducing to the statement over the complex numbers. This follows from complex uniformization, given that the connected Shimura datum are the same. For the second point, using the various functorialities of Shimura varieties, we clearly have a map of torsors $M_{dR}(B_1, X_{B_1}) \to \pi_1^\star M_{dR}(G_1, X_1) \times_{M_\mu^{ab,c}} \pi_2^\star M_{dR}(T, X_T)$ and any map of torsors is an isomorphism. For the third statement, we have a map $S^0(B_1, X_{B_1})_{g^{-1}(K)} \to S^0(G, X)_K$ by functoriality. To check that this is a finite étale map, we may use complex uniformization. The last point is again a consequence of the various functorial properties of Shimura varieties. □

In this paper, we will handle abelian type Shimura varities and construct several torsors over their toroidal compactifications following the strategy:

Strategy 4.54. (1) Extend all torsors from (G_1, X_1) to (B_1, X_{B_1}) using what we know on tori and the identity $M^c_{\mu_{B_1}} = M^c_{\mu_{G_1}} \times_{M^{ab,c}_{\mu_{G_1}}} T^c$.

(2) Descend the torsors from (B_1, X_1) to (G, X) by pushing out along the map $B_1^c \to G^c$ and take the quotient by a finite group.

A subtle point is that one must also check that the torsors statisfy the required equivariant properties giving the Hecke action. We insist that the stability under these extra structure does not follow from the above construction plan. In order to check that we have these extra structures, we also need to construct the perfectoid Shimura varieties in the abelian case, the Hodge-Tate period map, and prove several compatibilities. We do not need to work with perfectoid toroidal compactifications because one can check the various stabilities on the complement of the boundary. That this is possible is a consequence of the following lemma:

Lemma 4.55. *Let $\mathcal{G}$ be a locally of finite type group over $\mathrm{Spa}(F, \mathcal{O}_F)$. Let $\mathcal{G}^0 \hookrightarrow \mathcal{G}$ be a quasi-compact open subgroup. Let $\mathcal{X}$ be a finite type adic space over $\mathrm{Spa}(F, \mathcal{O}_F)$ and let $\mathcal{U} \hookrightarrow \mathcal{X}$ be a Zariski dense open. Let $\mathcal{T} \to \mathcal{X}$ be a $\mathcal{G}$-torsor. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two $\mathcal{G}_0$-reduction of this torsor. Assume that $\mathcal{T}_1|_{\mathcal{U}} = \mathcal{T}_2|_{\mathcal{U}}$. Then $\mathcal{T}_2 = \mathcal{T}_1$.*

Proof. We may assume that all torsors are trivial over $\mathcal{X}$. We see that there is an element $g \in \mathcal{G}(\mathcal{X})$ which realizes an isomorphism from $\mathcal{T}_1$ to $\mathcal{T}_2$. Moreover, by assumption, $g \in \mathcal{G}_0(\mathcal{U})$. We need to prove that $g \in \mathcal{G}_0(\mathcal{X})$. Consider the map $g : \mathcal{X} \to \mathcal{G}$. We claim that $g^{-1}(\mathcal{G}_0) = \mathcal{X}$. Note that $g^{-1}(\mathcal{G}_0)$ is quasi-compact and contains $\mathcal{U}$. Let $z \in \mathcal{X} \setminus g^{-1}(\mathcal{G}_0)$. We want to prove that some generization of z belongs to $g^{-1}(\mathcal{G}_0)$. Let $\{V_i\}$ be a decreasing sequence of quasi-compact neighborhoods of z such that $\cap_i V_i$ is the set of all generizations of z. Assume that $\cap_i V_i \cap g^{-1}(\mathcal{G}_0) = \emptyset$. Since $g^{-1}(\mathcal{G}_0)$ is compact in the constructible topology, we deduce that there is i such that $V_i \cap g^{-1}(\mathcal{G}_0) = \emptyset$. Therefore, $V_i \cap \mathcal{U} = \emptyset$ On the other hand, $\mathcal{U}$ is Zariski dense. A contradiction. It follows that $g^{-1}(\mathcal{G}_0)$ contains all maximal generizations of points of $\mathcal{X}$ and is constructible. By [Hub93], coro. 4.3, $\mathcal{X} = g^{-1}(\mathcal{G}_0)$. □

Remark 4.56. We admit that our treatment of abelian type Shimura varieties is a little ad hoc. The reason is that the following results are not yet available: the existence of nice and well described integral models for the compactifications, and the existence of perfectoid toroidal compactifications.

Remark 4.57. Perfectoid (minimal compactifications of) abelian type Shimura varieties have been constructed in [HJ20]. Since we also need to construct the Hodge-Tate period map and prove several compatibilities, we give a complete argument. The argument is very similar to [HJ20] in the sense that this is a reduction to the Hodge case.

We work until the end of this paragraph over the algebraically closed field $\mathbb{C} \simeq \mathbb{C}_p$. We first briefly recall how to reconstruct a Shimura variety from its connected Shimura variety following [Del79], section 2.1. Let (G, X) be a Shimura datum. We adopt the standard notations: $G^{ad}(\mathbb{Q})^+$ is the intersection of $G^{ad}(\mathbb{Q})$ with the neutral component of $G^{ad}(\mathbb{R})$, and $G(\mathbb{Q})_+$, $G^{der}(\mathbb{Q})_+$ are the inverse images of $G^{ad}(\mathbb{Q})^+$ via the natural morphisms $G^{der}(\mathbb{Q}) \to G(\mathbb{Q}) \to G^{ad}(\mathbb{Q})$. The center $Z(G)$ of G is simply denoted by Z unless some confusion may arise.

The inverse system $\lim_K S(G,X)_K = S(G,X)$ has a right action of the group $\mathcal{A}(G) = G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})} *_{G(\mathbb{Q})_+} G^{ad}(\mathbb{Q})^+$. Here $\overline{Z(\mathbb{Q})}$ is the closure of $Z(\mathbb{Q})$ in $G(\mathbb{A}_f)$.

Remark 4.58. If $Z_s = \{1\}$, then $\overline{Z(\mathbb{Q})} = Z(\mathbb{Q})$. If the morphism $G(\mathbb{Q}) \to G^{ad}(\mathbb{Q})$ is surjective (for example, if $Z(G)$ is a split torus), then $\mathcal{A}(G) = G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})}$.

There is a canonical bijection $\pi_0(S(G,X)) \to \overline{G(\mathbb{Q})_+}\backslash G(\mathbb{A}_f)$ of right $\mathcal{A}(G)$-profinite sets where $\overline{G(\mathbb{Q})_+}$ is the closure of $G(\mathbb{Q})_+$ in $G(\mathbb{A}_f)$. This is also the completion of $G(\mathbb{Q})_+$ for topology with basis of open of 1 given by the congruence subgroups of $G(\mathbb{Q})_+$.

Let us pick $1 \in \overline{G(\mathbb{Q})_+}\backslash G(\mathbb{A}_f)$ and let $S^0(G,X) \hookrightarrow S(G,X)$ be the connected component corresponding to 1. This is the tower of connected Shimura varieties. The stabilizor of $S^0(G,X)$ for the action of $\mathcal{A}(G)$ is $\mathcal{A}^0(G) = \overline{G(\mathbb{Q})}_+/\overline{Z(\mathbb{Q})} *_{G(\mathbb{Q})_+} G^{ad}(\mathbb{Q})^+$ and we have the formula:

$$S(G,X) = [S^0(G,X) \times \mathcal{A}(G)]/\mathcal{A}^0(G)$$

for the action $(s,g).h = (sh, h^{-1}g)$. One important observation is that $S^0(G,X)$ depends only on the connected Shimura datum (G^{der}, X^+) and the group $\mathcal{A}^0(G)$ depends also only on G^{der}. More precisely, we have that $\mathcal{A}^0(G) = \overline{G^{der}(\mathbb{Q})}_+ *_{G^{der}(\mathbb{Q})_+} G^{ad}(\mathbb{Q})^+$ and $\mathcal{A}^0(G)$ is therefore the completion of $G^{ad}(\mathbb{Q})^+$ with respect to the topology with basis of open of 1 the images of congruence subgroups of $G^{der}(\mathbb{Q})_+$.

We have that $S^0(G,X) = \lim_K S^0(G,X)/K$ where the limit runs over the compact open subgroups of $G^{der}(\mathbb{Q})_+$, and $(S^0(G,X)/K)(\mathbb{C}) = \Gamma_K\backslash X^+$ where $\Gamma_K = G^{der}(\mathbb{Q})_+ \cap K$.

Of course, there is also a simpler formula: $S(G,X) = [S^0(G,X)\times G(\mathbb{A}_f)]/\overline{G(\mathbb{Q})_+}$ but the disadvantage of this formula is that $\overline{G(\mathbb{Q})_+}$ depends on G ! Nonetheless, if we consider the de Rham torsor $M_{dR}(G,X)$ over $S(G,X)$, it does not carry an action $\mathcal{A}(G)$, but only a $G(\mathbb{A}_f)$-action. If we let $M^0_{dR}(G,X)$ be its restriction over $S^0(G,X)$, then we have the reconstruction formula: $M_{dR}(G,X) = [M^0_{dR}(G,X)\times G(\mathbb{A}_f)]/\overline{G(\mathbb{Q})_+}$. We thus see that the reconstruction formula for the de Rham torsor (with all its functorialities) from the connected component is slightly more delicate. We also observe that $M^0_{dR}(G,X)$ has a reduction of group structure to a $M^{c,der}_\mu$-torsor, but the equivariant structure depends on the M^c_μ-torsor.

Remark 4.59. There is also a reconstruction formula for the principal G^c-bundle, which uses a group $\mathcal{A}_{dR}(G)$ which fits in an exact sequence:

$$0 \to G^c \to \mathcal{A}_{dR}(G) \to \mathcal{A}(G) \to 0.$$

We will not use this, but remark that this formula uses and depends on G^c.

All this picture extends to the minimal compactification (except the last part concerning the de Rham torsor) since a Shimura variety and its minimal compactification have the same connected components and group actions extend by normality of the minimal compactifications.

We now go back to our Hodge datum (G_1,X_1) and we fix an embedding in a Siegel datum $(\tilde{G},\tilde{X})$. Attached to this fixed embedding, for any compact open K we have the minimal compactificaton $S^\star(G_1,X_1)_K$ together with a finite surjective map $S^\star(G_1,X_1)_{\overline{K}}$ where $S^\star(G_1,X_1)_{\overline{K}}$ is the schematic image of $S^\star(G_1,X_1)_K \to S^\star(\tilde{G},\tilde{X})_{\tilde{K}}$ for all small enough $\tilde{K}$ with $\tilde{K}\cap G(\mathbb{A}_f) = K$. Let us denote $\overline{S}^\star(G_1,X_1) = \lim_K S^\star(G_1,X_1)_{\overline{K}}$. For the Siegel datum $(\tilde{G},\tilde{X})$, we have that $\mathcal{A}(\tilde{G}) = G(\mathbb{A}_f)/Z(\mathbb{Q})$. We deduce that the closed subgroup $\mathcal{A}(G)$ of $\mathcal{A}(\tilde{G})$ acts on $\overline{S}^\star(G_1,X_1)$ and $S^\star(G_1,X_1)$ in a compatible way. We also deduce that the Hodge-Tate period map: $\mathcal{S}^\star(G_1,X_1) \to \overline{\mathcal{S}}^\star(G_1,X_1) \to \mathcal{FL}_{G_1,\mu_1}$ is $\mathcal{A}(G_1)$-equivariant (by reduction to the Siegel case). Remark that the $\mathcal{A}(G_1)$-action on $\mathcal{FL}_{G_1,\mu_1}$ factors through the map $\mathcal{A}(G_1) \to G_1^{ad}(\mathbb{A}_f) \to G_1^{ad}(\mathbb{Q}_p)$.

We are now ready to extend things to (B_1,X_{B_1}). First, we have that $S^\star(B_1,X_{B_1}) = [S^{\star,0}(G_1,X_1)\times\mathcal{A}(B_1)]/\mathcal{A}^0(G_1)]$. We may also define $\overline{S}^\star(B_1,X_{B_1}) = [\overline{S}^{\star,0}(G_1,X_1)\times \mathcal{A}(B_1)]/\mathcal{A}^0(G_1)]$. By taking K-invariants, we define $\overline{S}^\star(B_1,X_{B_1})/K = S^\star(B_1,X_{B_1})_{\overline{K}}$.

Our first lemma is:

Lemma 4.60. *There is a perfectoid space $\mathcal{S}^\star(B_1,X_{B_1})_{\overline{K^p}} \sim \lim_{K_p} \mathcal{S}^\star(B_1,X_{B_1})_{\overline{K_pK^p}}$ and a perfectoid space $\mathcal{S}^\star(B_1,X_{B_1})_{K^p} = \lim_{K_p} \mathcal{S}^{\star,\diamondsuit}(B_1,X_{B_1})_{K_pK^p}$.*

Proof. We prove the first statement. We see that K_p acts on the scheme $S^\star(B_1,X_{B_1})_{\overline{K^p}}$. It acts with finitely many orbits on the connected components. We choose representatives for the orbit, which are all of the form $S^{\star,0}(B_1,X_{B_1})_{k_i\overline{K^p}k_i^{-1}}$ for suitable elements $k_i \in B_1(\mathbb{A}_f)$. Let $S(k_i)$ be the stabilizor of the connected component $S^{\star,0}(B_1,X_{B_1})_{k_i\overline{K^p}k_i^{-1}}$ in K_p. This is a closed subgroup of K_p. We deduce that $S^\star(B_1,X_{B_1})_{\overline{K^p}} = \cup_i[S^{\star,0}(B_1,X_{B_1})_{k_i\overline{K^p}k_i^{-1}} \times K_p]/S(k_i)$. We now want to define $\mathcal{S}^\star(B_1,X_{B_1})_{\overline{K^p}} = \cup_i[\mathcal{S}^{\star,0}(B_1,X_{B_1})_{k_i\overline{K^p}k_i^{-1}} \times K_p]/S(k_i)$. To check that this is well defined, we remark that as an adic space (forgetting the action) each such space is the product $\mathcal{S}^{\star,0}(B_1, X_{B_1})_{k_i\overline{K^p}k_i^{-1}} \times (K_p/S(k_i))$ of a perfectoid space (by lemma 4.27 and the results of 4.4.7) with a profinite set. This is therefore a perfectoid space. Checking that $\mathcal{S}^\star(B_1, X_{B_1})_{\overline{K^p}} \sim \lim_{K_p} \mathcal{S}^\star(B_1, X_{B_1})_{\overline{K_pK^p}}$ is now an easy exercise using again 4.4.7. We deduce that $\mathcal{S}^\star(B_1, X_{B_1})_{K^p} = \lim_{K_p} \mathcal{S}^{\star,\diamond}(B_1, X_{B_1})_{K_pK^p}$ by [BS19], thm. 1.16. □

We can now construct the Hodge-Tate period map

$$\pi_{HT} : \mathcal{S}^\star(B_1, X_{B_1})_{K^p} \to \mathcal{S}^\star(B_1, X_{B_1})_{\overline{K^p}} \to \mathcal{FL}_{B_1,\mu_{B_1}}$$

by just composing the maps $\mathcal{S}^\star(B_1, X_{B_1})_{\overline{K^p}} \to \mathcal{S}^\star(G_1, X_1)_{\overline{K^p}} \to \mathcal{FL}_{G_1,\mu_1} = \mathcal{FL}_{B_1,\mu_{B_1}}$.

Lemma 4.61. *The Hodge-Tate period map is $B_1(\mathbb{Q}_p)$-equivariant and Hecke equivariant away from p. The map $\mathcal{S}^\star(B_1, X_{B_1})_{\overline{K^p}} \to \mathcal{FL}_{B_1,\mu_{B_1}}$ is affine.*

Proof. We may pass to the limit over K_p and consider $\overline{\mathcal{S}}^\star(B_1, X_{B_1}) = \overline{\mathcal{S}}^{\star,0}(G_1, X_1) \times \mathcal{A}(B_1)]/\mathcal{A}^0(G_1)] \to \mathcal{FL}_{B_1,\mu_{B_1}}$. Since the map $\overline{\mathcal{S}}^{\star,0}(G_1, X_1) \to \mathcal{FL}_{B_1,\mu_{B_1}}$ is $\mathcal{A}^0(G_1)$-equivariant, we obtain by extension a $\mathcal{A}(B_1)$-equivariant map: $\overline{\mathcal{S}}^\star(B_1, X_{B_1}) \to \mathcal{FL}_{B_1,\mu_{B_1}}$. The affine property follows from the Hodge case (G_1, X_1). □

We may now compare the Hodge-Tate and de Rham torsors.

Proposition 4.62. *Consider the perfectoid space $\mathcal{S}^{an}(B_1, X_{B_1})_{K^p}$ which is the complement of the boundary in $\mathcal{S}^\star(B_1, X_{B_1})_{K^p}$. Then there is a canonical isomorphism of $\mathcal{M}^{c,an}_{\mu_{B_1}}$-torsors between $\mathcal{M}^{an}_{HT}$ (pulled back via $\mathcal{S}^{an}(B_1, X_{B_1})_{K^p} \to \mathcal{FL}_{B_1,\mu_{B_1}}$) and $\mathcal{M}^{an}_{dR}$ (pulled back via $\mathcal{S}^{an}(B_1, X_{B_1})_{K^p} \to \mathcal{S}^{an}(B_1, X_{B_1})_{K_pK^p}$). This isomorphism if $B_1(\mathbb{Q}_p)$-equivariant and Hecke equivariant.*

Proof. The canonical isomorphism just follows from proposition **??** and 4.51, given theorem 4.53. □

We now consider the descent from (B_1, X_{B_1}) to (G, X). We have a map $f : B_1 \to G$ inducing an isogeny $B_1^{der} \to G^{der}$.

Lemma 4.63. (1) *We have a continuous surjective map $\mathcal{A}^0(B_1) \to \mathcal{A}^0(G)$ with kernel a finite group Δ.*

(2) *We have an étale morphism $S^0(B_1, X_{B_1}) \to S^0(G, X)$ with group Δ,*
(3) *For any neat compact open subgroup $K \in G^{der}(\mathbb{A}_f)$, we have a finite étale map:*
$$S^0(B_1, X_{B_1})/f^{-1}(K) \to S^0(G, X)/K$$
with group $\Delta(K) = (G^{der}(\mathbb{Q})_+ \cap K)/(B_1^{der}(\mathbb{Q})_+ \cap f^{-1}(K))$.
(4) *$\Delta = \lim_K \Delta(K)$ and the map $\Delta(K') \to \Delta(K)$ is injective if $K' \subseteq K$ are neat.*

Proof. We let $Cong(B_1)$ be the set of congruence subgroups of $B_1^{der}(\mathbb{Q})_+$ and $Cong(G)$ be the set of congruence subgroups of $G^{der}(\mathbb{Q}_+)$. We claim that the map $f : B_1^{der}(\mathbb{Q})^+ \to G^{der}(\mathbb{Q})^+$ induces a map $f^{-1} : Cong(G) \to Cong(B_1)$. Indeed, let $K \subseteq G^{der}(\mathbb{A}_f)$ be a compact open subgroup, then $f^{-1}(K)$ is compact open in $B_1^{der}(\mathbb{A}_f)$ and $f^{-1}(G^{der}(\mathbb{Q})^+ \cap K) = B_1^{der}(\mathbb{Q})^+ \cap f^{-1}K$. Since the maps $B_1^{der}(\mathbb{Q}_\ell) \to G^{der}(\mathbb{Q}_\ell)$ are local isomorphisms for all prime numbers ℓ, we deduce that the subset $f^{-1}(Cong(G))$ is cofinal in $Cong(B_1)$. If $\Gamma \in Cong(B_1)$, then $f(\Gamma)$ is an arithmetic subgroup of $G^{der}(\mathbb{Q})_+$ but not necessarily a congruence subgroup!

For any $\Gamma \in Cong(G)$, the group $f(f^{-1}(\Gamma))$ is a finite index, normal subgroup of Γ. Let $K \subseteq G^{der}(\mathbb{A}_f)$. We let $\Gamma_K = \Gamma \cap K$ and $\Delta(K) = \Gamma_K/f(f^{-1}(\Gamma_K))$. Let $g : G^{der} \to G^{ad}$ be the natural isogeny. We observe that for any sufficiently small $\Gamma \in Cong(B_1)$ or $\Gamma \in Cong(G)$, the map $f \circ g : \Gamma \to G^{ad}(\mathbb{Q})^+$ or $f : \Gamma \to G^{ad}(\mathbb{Q})^+$ is injective. Indeed, remark that the kernel of $G^{der}(\mathbb{Q})_+ \to G^{ad}(\mathbb{Q})^+$ is a finite group, and $\cap_{\Gamma \in Cong(G)}\Gamma = \{1\}$. The groups $g \circ f(Cong(B_1))$ and $f(Cong(G))$ are the basis of open neighborhoods of 1 for two topologies $\tau(B_1)$ and $\tau(G)$ on $G^{ad}(\mathbb{Q})_+$. We know that $\mathcal{A}^0(B_1)$ and $\mathcal{A}^0(G)$ are the completion of $G^{ad}(\mathbb{Q})^+$ for the topologies $\tau(B_1)$ and $\tau(G)$ respectively. Therefore, $\mathcal{A}^0(B_1) = \lim_{\Gamma \in Cong(B_1)} G^{ad}(\mathbb{Q})^+/\Gamma$ and $\mathcal{A}^0(G) = \lim_{\Gamma \in Cong(G)} G^{ad}(\mathbb{Q})^+/\Gamma$. By the Mittag-Leffler criterion the sequence

$$0 \to \lim_K \Delta(K) \to \lim_{f^{-1}(\Gamma_K)} G^{ad}(\mathbb{Q})^+/f^{-1}\Gamma_K \to \lim_{\Gamma_K} \mathcal{A}^0(G)/\Gamma_K \to 0$$

is exact and and writes $0 \to \Delta \to \mathcal{A}^0(B_1) \to \mathcal{A}^0(G) \to 0$. The point 3 follows by noting that $S^0(B_1, X_{B_1})/f^{-1}(K) \to S^0(G,X)/K$ has complex uniformization $f^{-1}\Gamma_K\backslash X^+ \to \Gamma_K\backslash X^+$ and the second point follows by passing to the limit. It is easy to check that if $K' \subseteq K$ are neat, $\Delta(K') \hookrightarrow \Delta(K)$. Therefore, Δ is a finite group. □

We are now ready to descend everything from (B_1, X_{B_1}) to (G,X). First, we have that $S^{\star,0}(G,X) = S^{\star,0}(B_1, X_{B_1})/\Delta$ and we may also define $\overline{S}^{\star,0}(G,X) = \overline{S}^{\star,0}(B_1, X_{B_1})/\Delta$. We have $S^\star(G,X) = [S^{\star,0}(G,X) \times \mathcal{A}(G)]/\mathcal{A}^0(G)$. We may also define $\overline{S}^\star(G,X) = [\overline{S}^{\star,0}(G,X) \times \mathcal{A}(G)]/\mathcal{A}^0(G)$. By taking K-invariants, we define $\overline{S}^\star(G,X)/K = S^\star(G,X)_{\overline{K}}$.

Theorem 4.64. *Let (G,X) be an abelian Shimura datum as above.*

(1) *There is a perfectoid space $\mathcal{S}^\star(G,X)_{\overline{K^p}} \sim \lim_{K_p} \mathcal{S}^\star(G,X)_{\overline{K_pK^p}}$ and a perfectoid space $\mathcal{S}^\star(G,X)_{K^p} = \lim_{K_p} \mathcal{S}^{\star,\diamondsuit}(G,X)_{K_pK^p}$.*
(2) *We have a Hodge-Tate period map $\pi_{HT} : \mathcal{S}^\star(G,X)_{K^p} \to \mathcal{S}^\star(G,X)_{\overline{K^p}} \to \mathcal{FL}_{G,\mu}$ which is $G(\mathbb{Q}_p)$-equivariant and Hecke equivariant. The map $\mathcal{S}^\star(G,X)_{\overline{K^p}} \to \mathcal{FL}_{G,\mu}$ is affine.*
(3) *There is a canonical isomorphism of $\mathcal{M}^{c,an}_{\mu_G}$-torsors between $\mathcal{M}^{an}_{HT}$ (pulled back via $\mathcal{S}^{an}(G,X)_{K^p} \to \mathcal{FL}_{G,\mu}$) and $\mathcal{M}^{an}_{dR}$ (pulled back via $\mathcal{S}^{an}(G,X)_{K^p} \to \mathcal{S}^{an}(G,X)_{K_pK^p}$). This isomorphism if $G(\mathbb{Q}_p)$-equivariant and Hecke equivariant.*

Proof. For the first point, we observe that the group $\Delta(K^p) = \lim_{K_p} \Delta(K_pK^p)$ acts trivially on the flag variety. Since π_{HT} was affinoid for (B_1, X_{B_1}), we are thus in a position to apply lemma 4.27 and lemma 4.28 to deduce that $\mathcal{S}^{\star,0}(G,X)_{\overline{K_p}}$ is perfectoid. We may now extend the result from $\mathcal{S}^{\star,0}(G,X)_{\overline{K_p}}$ to $\mathcal{S}^\star(G,X)_{\overline{K_p}}$ as in the proof of lemma 4.60.

Passing to the limit over K_p, we have a Hodge-Tate period map $\overline{\mathcal{S}}^{\star,0}(B_1, X_{B_1})/\Delta = \overline{\mathcal{S}}^{\star,0}(G,X) \to \mathcal{FL}_{G,\mu} = \mathcal{FL}_{B_1,\mu_{B_1}}$ which is $\mathcal{A}(G)^0$-equivariant. We deduce that there is a $\mathcal{A}(G)$-equivariant Hodge-Tate period map $\overline{\mathcal{S}}^\star(G,X) \to \mathcal{FL}_{G,\mu}$. The representability of $\lim_{K_p} \mathcal{S}^{\star,\diamondsuit}(G,X)_{K_pK^p}$ follows from [BS19], thm. 1.16.

For the last point, we first pass to the limit over all K to turn the Hecke action into a $G(\mathbb{A}_f)$-action. We see that the two torsors are canonically isomorphic over $\mathcal{S}^{an,0}(G,X)$ by descent from $\mathcal{S}^{an,0}(B_1, X_{B_1})$ and proposition 4.62. We now need to check that the $\overline{G(\mathbb{Q})_+}$ action match. Granting the isomorphism of the torsors, we see that the difference between the two $\overline{G(\mathbb{Q})_+}$-actions is given by a continuous map $\overline{G(\mathbb{Q})_+} \to M^c_\mu(\mathcal{S}^{an,0}(G,X))$. This map is semi-linear with respect to the action of $\overline{G(\mathbb{Q})_+}$ on $\mathrm{H}^0(\mathcal{S}^{an,0}(G,X), \mathscr{O}_{\mathcal{S}^{an,0}(G,X)})$. It therefore gives a class in $\mathrm{H}^1(\overline{G(\mathbb{Q})_+}, M^c_\mu(\mathcal{S}^{an,0}(G,X)))$. We wan to show that this class is trivial. It is certainly trivial when restricted to $\mathrm{Im}(\overline{B_1(\mathbb{Q})_+} \to \overline{G(\mathbb{Q})_+}) = \Gamma$. We have the short exact sequence of pointed sets in non-abelian cohomology :

$$1 \to \mathrm{H}^1(\overline{G(\mathbb{Q})_+}/\Gamma, (M^c_\mu(\mathcal{S}^{an,0}(G,X)))^\Gamma) \to \mathrm{H}^1(\overline{G(\mathbb{Q})_+}, M^c_\mu(\mathcal{S}^{an,0}(G,X)))$$
$$\to \mathrm{H}^1(\Gamma, M^c_\mu(\mathcal{S}^{an,0}(G,X)))$$

We claim that $\mathrm{H}^0(\mathcal{S}^{an,0}(G,X), \mathscr{O}_{\mathcal{S}^{an,0}(G,X)})^\Gamma = \mathbb{C}_p$ and therefore that

$$(M^c_\mu(\mathcal{S}^{an,0}(G,X)))^\Gamma = M^c_\mu(\mathbb{C}_p).$$

It suffices to check that $\mathrm{H}^0(\mathcal{S}^{an,0}(B_1, X_{B_1}), \mathscr{O}_{\mathcal{S}^{an,0}(B_1,X_{B_1})})^{\overline{B_1(\mathbb{Q})_+}} = \mathbb{C}_p$. To prove this, first observe that the invariant are functions on some finite level connected Shimura variety by [Sch13a], coro. 6.19, and then use Zariski density of Hecke orbits. It follows that the cohomology class is actually given by a group homomorphism $\overline{G(\mathbb{Q})_+} \to M^c_\mu(\mathbb{C}_p)$, trivial on Γ. By continuity, it suffices to check that this group homomorphism is trivial on $G(\mathbb{Q})_+$. In order to conclude, we can consider special Shimura varieties $(H, X_H) \to (G, X)$. By functoriality, we get that the pull back of the two torsors is canonically isomorphic with their $H(\mathbb{A}_f)$-action over $\mathcal{S}(H, X_H)$ by lemma 4.51. Since $G(\mathbb{Q})_+$ is generated by the $G(\mathbb{Q})_+ \cap H(\mathbb{Q})$ where (H, X_H) runs over all special Shimura varieties by [Har85], lemma 3.13.1, we conclude that the map $\overline{G(\mathbb{Q})_+} \to M^c_\mu(\mathbb{C}_p)$ is trivial.

□

4.5. **Affiness of the truncated Hodge-Tate period map.** In most of this paper, we work on finite level Shimura varieties rather than perfectoid Shimura varieties. For this reason we introduce some truncated Hodge-Tate period map.

Let (G,X) be an abelian Shimura datum. Let $K = K_pK^p \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. The group K_p acts on $\mathcal{FL}_{G,\mu}$. We form the quotient space $\mathcal{FL}_{G,\mu}/K_p$, equipped with the quotient topology from the surjective map $\pi_{K_p} : \mathcal{FL}_{G,\mu} \to \mathcal{FL}_{G,\mu}/K_p$. We merely view $\mathcal{FL}_{G,\mu}/K_p$ as a topological space. We adopt some definitions. We say that an open $U \subseteq \mathcal{FL}_{G,\mu}/K_p$ is affinoid if $\pi^{-1}_{K_p}(U)$ is affinoid. If $V \subseteq U \subseteq \mathcal{FL}_{G,\mu}/K_p$ are open we say that V is a rational subset of U if $\pi^{-1}_{K_p}(V)$ is a rational subset of $\pi^{-1}_{K_p}(U)$. We let the residue field of a point $x \in \mathcal{FL}_{G,\mu}/K_p$ be the residue field of any lift of this point to $\mathcal{FL}_{G,\mu}$. To illustrate all these definitions, we have the following lemma:

Lemma 4.65. *Any point $x \in \mathcal{FL}_{G,\mu}/K_p$ with finite residue field over $\mathbb{Q}_p$ (i.e. a classical point in the sense of rigid analytic geometry) has a basis of neighborhoods consisting of affinoids $\{U_n\}_{n\geq 1}$ with the property that $U_{n+1} \subseteq U_n$ is a rational subset.*

Proof. Since G splits over F, it admits a reductive a model over $\mathcal{O}_F$. We let $\mathcal{G}$ be the (quasi-compact) adic space attached to the p-adic completion of this reductive model. The group $\mathcal{G}$ admits a filtration by normal open affinoid subgroups $\mathcal{G}_n$ which form a basis of neighborhood of the identity (take $\mathcal{G}_n$ the subgroup of elements which reduce to 1 modulo p^n). Let $y \in \mathcal{FL}_{G,\mu}$ be a lift of x. Then $y\mathcal{G}_n \hookrightarrow \mathcal{FL}_{G,\mu}$ is an affinoid for any $n \geq 1$ (this is a closed tube centered at the point y, this is also where we use that x has finite residue field over $\mathbb{Q}_p$). We now consider the group $\mathcal{G}_n K_p \subset \mathcal{G}$. The connected component of the identity of this group is $\mathcal{G}_n$, and the quotient $\mathcal{G}_n K_p/\mathcal{G}_n$ is a finite group. Then we find that $y\mathcal{G}_n K_p = \coprod_{i \in I} yk_i\mathcal{G}_n$ for a finite set I and elements $k_i \in K_p$, and is therefore an affinoid which is K_p-invariant. The $\{y\mathcal{G}_n K_p/K_p\}_{n\geq 1}$ form a basis of open neighborhoods of x. Moreover, $y\mathcal{G}_n K_p \subseteq y\mathcal{G}_{n-1}K_p$ is a rational subset because $yk_i\mathcal{G}_n \subseteq yk_i\mathcal{G}_{n-1}$ is a rational subset. □

The main result of this section is the following:

Theorem 4.66. *There is a continuous map:*

$$\pi_{HT,K_p} : \mathcal{S}^\star_K \to \mathcal{FL}_{G,\mu}/K_p$$

which is equivariant for the action of the Hecke algebra $\mathcal{C}^\infty_c(K\backslash G(\mathbb{A}_f)/K, \mathbb{Z})$ by correspondences.

Moreover, any point $x \in \mathcal{FL}_{G,\mu}/K_p$ with finite residue field over $\mathbb{Q}_p$ (i.e. a classical point in the sense of rigid analytic geometry) has an affinoid neighborhood U such that for any rational subset $V \subseteq U$, $(\pi_{HT,K_p})^{-1}(V)$ is affinoid.

By theorem 4.64 there is a map $\pi_{HT} : \mathcal{S}^\star_{\overline{K^p}} \to \mathcal{FL}_{G,\mu}$ which is equivariant for the action of the $G(\mathbb{Q}_p)$. For any point $x \in \mathcal{FL}_{G,\mu}$, there is an affinoid neighborhood U of x such that $\pi^{-1}_{HT}(U)$ is affinoid. Moreover, $\pi^{-1}_{HT}(U) = \lim_{K_p} \pi^{-1}_{HT}(U)_{K_p}$ where for K_p small enough, $\pi^{-1}_{HT}(U)_{K_p} \hookrightarrow \mathcal{S}^\star_{\overline{K_pK^p}}$ is affinoid. We call an open affinoid in $\mathcal{FL}_{G,\mu}$ with these properties a very good affinoid. Clearly, a rational subset of a very good affinoid is a very good affinoid.

We can define truncated Hodge-Tate period maps:

$$\pi_{HT,K_p} : \mathcal{S}^\star_{K_pK^p} \to \mathcal{S}^\star_{\overline{K_pK^p}} \to \mathcal{FL}_{G,\mu}/K_p.$$

Lemma 4.67. *The map π_{HT,K_p} is continuous.*

Proof. We have a continuous map (of topological spaces) $\lim_{K'_p} \mathcal{S}^\star_{K'_pK^p} \to \mathcal{S}^\star_{\overline{K^p}} \to \mathcal{FL}_{G,\mu}$. For all normal subgroup $K'_p \subseteq K_p$, the map $\mathcal{S}^\star_{K'_pK^p} \to \mathcal{S}^\star_{K_pK^p}$ is surjective and the target carries the quotient topology ([Han19], theorem 1.1). The continuity of the map of the lemma follows. □

Lemma 4.68. *Let $V \subseteq \mathcal{FL}_{G,\mu}$ be a very good affinoid invariant under a compact open subgroup K_p. Let $\overline{V}$ be its image in $\mathcal{FL}_{G,\mu}/K_p$. Then $\pi^{-1}_{HT,K_p}(\overline{V}) \subseteq \mathcal{S}^\star_{K_pK^p}$ is affinoid.*

Proof. It is part of the definition that $\pi^{-1}_{HT}(V)$ is the pullback of an affinoid $\pi^{-1}_{HT}(V)_{K'_p} \subset \mathcal{S}^\star_{\overline{K'_pK^p}}$ for some $K'_p \subseteq K_p$, a normal compact open subgroup. We can further pull back $\pi^{-1}_{HT}(V)_{K'_p}$ to an affinoid $\tilde{U} \subseteq \mathcal{S}^\star_{K'_pK^p}$. The space $\mathcal{S}^\star_{K_pK^p}$ is the categorical quotient of $\mathcal{S}^\star_{K'_pK^p}$ by K_p/K'_p and the image of $\tilde{U}$ in $\mathcal{S}^\star_{K_pK^p}$ is indeed affinoid. □

Proof of Theorem 4.66. The continuity of the map is lemma 4.67. Let $x \in \mathcal{FL}_{G,\mu}/K_p$ with finite residue field over $\mathbb{Q}_p$. Let y be a lift of x in $\mathcal{FL}_{G,\mu}$. For n large enough, $x\mathcal{G}_n \hookrightarrow \mathcal{FL}_{G,\mu}$ is a good affinoid (where $\mathcal{G}_n$ is the subgroup of elements which reduce to 1 modulo p^n as in lemma 4.65). We form the group $\mathcal{G}_n K_p$. Then $y\mathcal{G}_n K_p = \coprod_{i\in I} xk_i\mathcal{G}_n$ for a finite set I and elements $k_i \in K_p$. Moreover, $yk_i\mathcal{G}_n = y\mathcal{G}_n k_i$ is a very good affinoid (because the property of being a very good affinoid is preserved under $G(\mathbb{Q}_p)$-action). A finite disjoint union of very good affinoids is a very good affinoid. We can apply lemma 4.68, to $y\mathcal{G}_n K_p$. The image of $y\mathcal{G}_n K_p$ in $\mathcal{FL}_{G,\mu}/K_p$ provides the open neighborhood U of x as in the theorem. Finally, since any rational subset of a very good affinoid is again a very good affinoid, a second application of lemma 4.68 proves the last point. ☐

We also adopt the notation $\pi^{tor}_{HT,K_p} : \mathcal{S}^{tor}_{K_pK^p,\Sigma} \to \mathcal{S}^{\star}_{K_pK^p} \stackrel{\pi_{HT,K_p}}{\to} \mathcal{FL}_{G,\mu}/K_p$.

4.6. Reduction of the group structure of the torsor $\mathcal{M}^{an}_{dR}$.

The group G is defined over $\mathbb{Q}_p$. We have chosen a finite extension F of $\mathbb{Q}_p$ which splits G. We have also fixed a representative of the cocharacter μ over F and let P_μ and M_μ be the corresponding parabolic and Levi subgroups of G. We will soon assume that G is quasi-split over $\mathbb{Q}_p$. When this is the case we assume that P_μ contains a Borel B defined over $\mathbb{Q}_p$. We take a reductive model for G_F defined over Spec $\mathcal{O}_F$. By abuse of notation, we keep denoting this model by G. We also have models for P_μ and M_μ over Spec $\mathcal{O}_F$. On the analytic side, we have the (non-quasi-compact) groups $\mathcal{G}^{an}$, $\mathcal{P}^{an}_\mu$, $\mathcal{M}^{an}_\mu$ all considered over $\mathrm{Spa}(F, \mathcal{O}_F)$, and there is an embedding $G(\mathbb{Q}_p) \hookrightarrow \mathcal{G}^{an}$. Because we have fixed an integral model for G, we also have the quasi-compact groups $\mathcal{G} \hookrightarrow \mathcal{G}^{an}$, $\mathcal{P}_\mu \hookrightarrow \mathcal{P}^{an}_\mu$, $\mathcal{M}_\mu \hookrightarrow \mathcal{M}^{an}_\mu$.

The goal of this section is to prove that using the Hodge-Tate period morphism, we can produce some finer structure on the torsor $\mathcal{M}^{an}_{dR}$. These result generalize those obtained in [AIP15], prop. 4.3.1 for example. These refined structure will allow us to p-adically interpolate automorphic vector bundles.

4.6.1. *Preparations.* We first consider a Hodge datum (G, X) and let Σ be a perfect cone decomposition. We start by giving a detailed description of the torsor arising from the map:

$$\pi^{tor}_{HT} : \mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{FL}_{G,\mu}.$$

Let $S \to \mathcal{S}^{tor}_{K^p,\Sigma}$ be a map from a perfectoid space S. By composing with π^{tor}_{HT}, we get a map $S \to \mathcal{FL}_{G,\mu}$ which can be described as follows. Over S, we have a map $\mathcal{P}^{an}_{HT} \hookrightarrow \mathcal{G}^{an} \times S$, from the $\mathcal{P}^{an}_\mu$-torsor $\mathcal{P}^{an}_{HT}$ to the trivial $\mathcal{G}^{an}$-torsor, which is equivariant for the natural morphism of groups $\mathcal{P}^{an}_\mu \to \mathcal{G}^{an}$.

There is an étale cover $\tilde{S} \to S$ such that the torsor $\mathcal{P}^{an}_{HT} \times_S \tilde{S}$ becomes trivial and adquires a section $g_{\tilde{S}} \in \mathcal{G}^{an}(\tilde{S})$, unique up to left multiplication by elements of $\mathcal{P}^{an}_\mu(\tilde{S})$ so that we get a commutative diagram of torsors:

$$\begin{array}{ccc} \mathcal{P}^{an}_{HT} \times_S \tilde{S} & \longrightarrow & \mathcal{G}^{an} \times \tilde{S} \\ \downarrow \wr & & \downarrow \mathrm{Id} \\ (\mathcal{P}^{an}_\mu \times \tilde{S}).g_{\tilde{S}} & \longrightarrow & \mathcal{G}^{an} \times \tilde{S} \end{array}$$

Over $\tilde{S} \times_S \tilde{S}$, there is a section $h_{\tilde{S}\times_S\tilde{S}} \in \mathcal{P}_\mu^{an}(\tilde{S} \times_S \tilde{S})$ such that $h_{\tilde{S}\times_S\tilde{S}}.p_1^\star g_{\tilde{S}} = p_2^\star g_{\tilde{S}}$.

The class $h_{\tilde{S}\times_S\tilde{S}}$ is a 1-cocycle which describes the original torsor $\mathcal{P}_{HT}^{an}$. Changing $g_{\tilde{S}}$ by left multiplication by an element of $\mathcal{P}_\mu^{an}(\tilde{S})$ will change $h_{\tilde{S}\times_S\tilde{S}}$ by a coboundary.

The image of $g_{\tilde{S}} \in \mathcal{FL}_{G,\mu}(\tilde{S})$ descends to give a point in $\mathcal{FL}_{G,\mu}(S)$: the morphism $S \to \mathcal{FL}_{G,\mu}$ we started with.

The group $\mathcal{G}^{an}$ acts on the right on $\mathcal{FL}_{G,\mu}$. Concretely this action sends $g_{\tilde{S}}$ to $g_{\tilde{S}}.g$. This action does not affect the construction of the torsor $\mathcal{P}_{HT}^{an}$ wich is indeed $\mathcal{G}^{an}$-equivariant.

4.6.2. *Integral structure.* Let (G, X) be an abelian datum. Recall that $\mathcal{M}_\mu$ is (a quasi-compact) open subgroup of $\mathcal{M}_\mu^{an}$. The following proposition can be interpreted as the existence of an integral structure on the torsors $\mathcal{M}_{dR}^{an}$ or $\mathcal{M}_{HT}^{an}$.

Proposition 4.69. *Let $K_p \subset G(\mathbb{Q}_p)\cap G(\mathcal{O}_F)$. The étale torsor $\mathcal{M}_{dR}^{an} = \mathcal{M}_{HT}^{an}$ over $\mathcal{S}^{tor}_{K^pK_p,\Sigma}$ has a reduction of structure group to an étale $\mathcal{M}_\mu^c$-torsor $\mathcal{M}_{dR} = \mathcal{M}_{HT}$.*

Remark 4.70. In the Siegel case, the torsor $\mathcal{M}_{dR}^{an}$ is (ignoring the center) the torsor of trivializations of the vector bundle ω_A, the conormal sheaf of the universal semi-abelian scheme over $\mathcal{S}^{tor}_{K^pK_p,\Sigma}$ (well defined up to prime to p-isogeny by our choice of level structure). A possible integral structure is obtained by declaring that an invariant differential form is integral if it extends to an invariant differential form on an integral model of the universal semi-abelian scheme. However, the integral structure we consider here is different. Namely, we declare that a differential form is integral if it is in the span of the image of the integral Tate module for the Hodge-Tate period map. By [Far11], Theorem 2, section 5.3.2. we can explicitly bound the difference between both integral structures.

Proof. We first consider the Hodge case and take a perfect cone decomposition Σ. We work over $\mathcal{S}^{tor}_{K^p,\Sigma}$. Let $S \to \mathcal{S}^{tor}_{K^p,\Sigma}$ be a map from a perfectoid space. We first explain how to define the torsor $\mathcal{M}_{HT}$ over S (in a functorial way).

The map $S \to \mathcal{FL}_{G,\mu}$ is described by an element $g_{\tilde{S}} \in \mathcal{G}^{an}(\tilde{S})$ for some étale cover $\tilde{S} \to S$. We are free to change $g_{\tilde{S}}$ by left multiplication by an element of $\mathcal{P}_\mu^{an}(\tilde{S})$. Thus, up to passing to some further cover of $\tilde{S}$, we may actually assume that $g_{\tilde{S}} \in \mathcal{G}(\tilde{S})$ (because $\mathcal{FL}_{G,\mu} = \mathcal{P}_\mu\backslash\mathcal{G} = \mathcal{P}_\mu^{an}\backslash\mathcal{G}^{an}$), and this new element is well defined up to multiplication by an element of $\mathcal{P}_\mu(\tilde{S})$. The torsor associated is defined by the cocycle $p_2^\star g_{\tilde{S}}.(p_1^\star g_{\tilde{S}})^{-1} = h_{\tilde{S}\times_S\tilde{S}} \in \mathcal{P}_\mu(\tilde{S} \times_S \tilde{S}) \subset \mathcal{P}_\mu^{an}(\tilde{S} \times_S \tilde{S})$. We therefore have produced a reduction of the torsor $\mathcal{P}_{HT}^{an}$ to a torsor $\mathcal{P}_{HT}$ under the group $\mathcal{P}_\mu$. We can take the pushout under the map $\mathcal{P}_\mu \to \mathcal{M}_\mu$ (which amounts to projecting $h_{\tilde{S}\times_S\tilde{S}}$ in $\mathcal{M}_\mu(\tilde{S} \times_S \tilde{S})$) to get the desired torsor $\mathcal{M}_{HT}$.

We now proceed to descend from $\mathcal{S}^{tor}_{K^p,\Sigma}$ to $\mathcal{S}^{tor}_{K^pK_p,\Sigma}$. We have an étale torsor $\mathcal{M}_{HT}^{an} \to \mathcal{S}^{tor}_{K^pK_p,\Sigma}$ and we have defined an open subset $\mathcal{M}_{HT} \subset \mathcal{M}_{HT}^{an} \times_{\mathcal{S}^{tor}_{K^pK_p,\Sigma}} \mathcal{S}^{tor}_{K^p,\Sigma}$. We claim that this open descends to an open subset of $\mathcal{M}_{dR}^{an}$. The map $\mathcal{M}_{HT}^{an} \times_{\mathcal{S}^{tor}_{K,\Sigma}} \mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{M}_{HT}^{an}$ identifies the topological space $|\mathcal{M}_{HT}^{an}|$ with the quotient of $|\mathcal{M}_{HT}^{an} \times_{\mathcal{S}^{tor}_{K,\Sigma}} \mathcal{S}^{tor}_{K^p,\Sigma}|$ for the action of K_p. We have an identification between K_p-invariant open subsets of $\mathcal{M}_{HT}^{an} \times_{\mathcal{S}^{tor}_{K,\Sigma}} \mathcal{S}^{tor}_{K^p,\Sigma}$ and open subsets of $\mathcal{M}_{HT}^{an}$. The only thing to check is therefore that $\mathcal{M}_{HT}$ is indeed invariant under the action of K_p.

We go back to considering a map $S \to \mathcal{S}^{tor}_{K^p,\Sigma}$, described by an element $g_{\tilde{S}} \in \mathcal{G}(\tilde{S})$. Under right multiplication by $k \in K_p$, we get a new element $g_{\tilde{S}}k \in \mathcal{G}(\tilde{S})$ (it is crucial here that $k \in K_p \subseteq G(\mathcal{O}_F)$) and the corresponding reduction of the torsor described by the element $p_2^\star g_{\tilde{S}}k.(p_1^\star g_{\tilde{S}}.k)^{-1} = p_2^\star g_{\tilde{S}}.(p_1^\star g_{\tilde{S}})^{-1}$ doesn't depend on k.

We need to prove that the action map $\mathcal{M}^{an}_\mu \times \mathcal{M}^{an}_{HT} \to \mathcal{M}^{an}_{HT} \times \mathcal{M}^{an}_{HT}$ (which is an isomorphism), induces an isomorphism $\mathcal{M}_\mu \times \mathcal{M}_{HT} \to \mathcal{M}_{HT} \times \mathcal{M}_{HT}$. In other words, we need to prove that the two open subsets $\mathcal{M}_\mu \times \mathcal{M}_{HT}$ and $\mathcal{M}_{HT} \times \mathcal{M}_{HT}$ identify via the action map. This can be checked after pull back to $S^{tor}_{K^p,\Sigma}$ and this is true. Finally, the morphism $\mathcal{M}_{HT} \to S^{tor}_{K,\Sigma}$ is smooth, and surjective on geometric points. Therefore, there are sections étale-locally and $\mathcal{M}_{HT}$ is an étale torsor.

We remark that we may now change the cone decomposition and use pull back to define the torsor on a non perfect cone decomposition.

We now do the extension to the abelian case using the strategy 4.54. We consider a diagram of Shimura datum:

$$\begin{array}{ccc} (B_1, X_{B_1}) & \longrightarrow & (B, X_B) \\ \downarrow & & \downarrow \\ (G_1, X_1) & & (G, X) \end{array}$$

We may assume that the various morphisms between the groups G_1, B_1, B, G over $\mathbb{Q}$ extend to morphisms over $\mathcal{O}_F$. We first go from G_1 to B_1. We have a map $\pi_1 : \mathcal{S}^{tor}(B_1, X_{B_1})_{K,\Sigma} \to \mathcal{S}^{tor}(G_1, X_1)_{K_1,\Sigma_1}$ and a map $\pi_2 : \mathcal{S}^{tor}(B_1, X_{B_1})_{K,\Sigma} \to \mathcal{S}(T, X_T)_{K_2}$. For suitable compact open subgroups. We can push-out the $K^c_{2,p}$-torsor given by the tower of Shimura varieties along the map $K_{2,p} \to \mathcal{T}^c$ (where $\mathcal{T}$ is the quasi-compact group attached to $T \to \mathrm{Spec}\ \mathcal{O}_F$), to get a torsor $\mathcal{T}_{HT}$. The product $\pi_1^\star \mathcal{M}_{HT} \times_{M^{ab}_{\mu_{G_1}}} \mathcal{T}_{HT}$ gives a $\mathcal{M}^c_{\mu_{B_1}}$-torsor. We can then descend from (B_1, X_{B_1}) to (G, X). We consider a connected component $\mathcal{S}^{tor,0}(G, X)_{K,\Sigma}$. We have a finite flat map $\mathcal{S}^{tor,0}(B_1, X_{B_1})_{g^{-1}K,\Sigma} \to \mathcal{S}^{tor,0}(G, X)_{K,\Sigma}$. We consider the pushout $\mathcal{M}_{HT} \times^{M^c_{\mu_{B_1}}} M^c_{\mu_G}$ and we claim that this torsor descends. Note that away from the boundary, we can give a direct construction (using the Hodge-Tate period map as above). The descent data and all the functorialities extend by lemma 4.55. □

Remark 4.71. The above argument uses crucially that we know that $\mathcal{M}^{an}_{HT}$ descends from $S^{tor}_{K^p,\Sigma}$ to $S^{tor}_{K,\Sigma}$, because pro-étale descent is not effective in general.

For $\kappa \in X^\star(T^c)^{M_\mu,+}$, the sheaf $\mathcal{V}_\kappa$ is modeled on the representation V_κ of $\mathcal{M}^{an}_\mu$ defined over F. Recall that V_κ is defined as the module of sections $f(m) \in \mathrm{H}^0(\mathcal{M}^{an}_\mu, \mathscr{O}_{\mathcal{M}^{an}_\mu})$ such that $f(mb) = -w_{0,M}\kappa(b)f(m)$ for $b \in \mathcal{B}^{an} \cap \mathcal{M}^{an}_\mu$. Using that

$$\mathcal{M}^{an}_\mu/(\mathcal{B}^{an} \cap \mathcal{M}^{an}_\mu) = \mathcal{M}_\mu/(\mathcal{B} \cap \mathcal{M}_\mu)$$

we find that this is also the module of sections $f(m) \in \mathrm{H}^0(\mathcal{M}_\mu, \mathscr{O}_{\mathcal{M}_\mu})$ such that $f(mb) = -w_{0,M}\kappa(b)f(m)$ for $b \in \mathcal{B} \cap \mathcal{M}_\mu$.

We can define an $\mathcal{O}_F$-submodule $V^+_\kappa \subset V_\kappa$ by considering sections $f(m) \in \mathrm{H}^0(\mathcal{M}_\mu, \mathscr{O}^+_{\mathcal{M}_\mu})$ such that $f(mb) = -w_{0,M_\mu}\kappa(b)f(m)$. Since $\mathrm{H}^0(\mathcal{M}_\mu, \mathscr{O}^+_{\mathcal{M}_\mu})$ is open and bounded in $\mathrm{H}^0(\mathcal{M}_\mu, \mathscr{O}_{\mathcal{M}_\mu})$, we deduce that V^+_κ is a lattice in V_κ, stable under the action of $\mathcal{M}_\mu$.

Corollary 4.72. *For all $\kappa \in X^\star(T^c)^{M_\mu,+}$, the locally free sheaf $\mathcal{V}_\kappa$ over $\mathcal{S}^{tor}_{K,\Sigma}$ has an integral structure $\mathcal{V}^+_\kappa$ in the sense of definition 2.26.*

Proof. We consider the map $g : \mathcal{M}_{dR} \to \mathcal{S}^{tor}_{K,\Sigma}$. We let $\mathcal{V}^+_\kappa$ be the subsheaf of $g_\star \mathscr{O}^+_{\mathcal{M}_{dR}}$ of sections $f(m)$ which satisfy $f(mb) = -w_{0,M}\kappa(b)f(m)$. □

4.6.3. *Further reductions of the group structure.* We assume that $K_p = K_{p,m',b'}$ for $m' \geq b' \in \mathbb{Z}_{\geq 0}$ and $m' > 0$ (see section 3.5.1). This is a compact open subgroup of $G(\mathbb{Q}_p)$ with an Iwahori decomposition. For any $w \in {}^MW$, we let K_{p,w,M_μ} be the projection of $wK_pw^{-1} \cap \mathcal{P}_\mu$ to $\mathcal{M}_\mu$.

We will describe this group. We can define $N_{p,w,M_\mu} = K_{p,w,M_\mu} \cap U_{M_\mu}$ and $\overline{N}_{p,w,M_\mu} = K_{p,w,M_\mu} \cap \overline{U}_{M_\mu}$, where we let B_{M_μ} be the Borel subgroup of M_μ which is the image of B in M_μ, $\overline{B}_{M_\mu}$ the opposite Borel, and U_{M_μ} (resp. $\overline{U_{M_\mu}}$) be the unipotent radical of B_{M_μ} (resp. $\overline{B}_{M_\mu}$). We also recall that $T_{b'} = \mathrm{Ker}(T(\mathcal{O}_F) \to T(\mathcal{O}_F/\varpi^{b'}\mathcal{O}_F)) \cap T(\mathbb{Q}_p) = T \cap K_p$.

Proposition 4.73. *For any $w \in {}^MW$, the group K_{p,w,M_μ} is a subgroup of the Iwahori subgroup of $M_\mu(\mathcal{O}_F)$. Moreover, it admits an Iwahori decomposition. Namely, the product map:*

$$N_{p,w,M_\mu} \times wT_{b'}w^{-1} \times \overline{N}_{p,w,M_\mu} \to K_{p,w,M_\mu}$$

is an isomorphism.

When G is unramified, M_μ is defined over $\mathbb{Q}_p$, w is $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariant, and $K_p = K_{p,1,0}$ is the Iwahori subgroup of $G(\mathbb{Z}_p)$, then K_{p,w,M_μ} is the Iwahori subgroup of $M_\mu(\mathbb{Z}_p)$.

Proof. Let U and $\overline{U}$ be respectively the unipotent radicals of B and $\overline{B}$. We have the Iwahori decomposition $K_p = \overline{N}_p \times T_{b'}(\mathbb{Z}_p) \times N_p$ where $N_p = K_p \cap U$ is $U(\mathbb{Q}_p) \cap U(\mathcal{O}_F)$ and $\overline{N}_p = K_p \cap \overline{U} \subseteq \overline{U}(\mathbb{Q}_p)$ is the subgroup of elements reducing to 1 modulo $\varpi^{m'}$ It is useful to give a more precise version of this decomposition. We let Φ be the set of roots (defined over F). We have $\Phi = \Phi^+_M \coprod \Phi^-_M \coprod \Phi^{+,M} \coprod \Phi^{-,M}$, where $\Phi_M = \Phi^+_M \coprod \Phi^-_M$ is the set of roots in M. We also let $\Phi_0 = \Phi/\mathrm{Gal}(F/\mathbb{Q}_p)$ and have $\Phi_0 = \Phi^+_0 \coprod \Phi^-_0$ (because G is quasi-split).

For all $\alpha_0 \in \Phi_0$, we let $U_{\alpha_0} \hookrightarrow G$ be the corresponding unipotent group. For all $\alpha \in \Phi$ we also denote by $U_\alpha \hookrightarrow G_{\mathcal{O}_F}$ the one parameter subgroup. We have $U_{\alpha_0} \times_{\mathrm{Spec}\ \mathbb{Q}_p} \mathrm{Spec}\ F = \prod_{\alpha \in \Phi, \alpha \mapsto \alpha_0} U_\alpha \times_{\mathrm{Spec}\ \mathcal{O}_F} \mathrm{Spec}\ F$. We consider the product map (in any order of the factors):

$$\prod_{\alpha_0} U_{\alpha_0}(\mathbb{Q}_p) \times T(\mathbb{Q}_p) \to G(\mathbb{Q}_p).$$

This maps induces a bijection between K_p and the set of elements $((n_{\alpha_0})_{\alpha_0 \in \Phi_0}, t)$ which satisfy: $n_{\alpha_0}, t \in G(\mathcal{O}_F)$, $n_{\alpha_0} = 1 \mod \varpi^{m'}$ if $\alpha_0 \in \Phi^-_0$ and $t = 1 \mod \varpi^{b'}$. We get that $wK_pw^{-1} \cap P_\mu$ identifies (via the product map) with the set of elements $((wn_{\alpha_0}w^{-1})_{\alpha_0 \in \Phi_0} \in wU_{\alpha_0}(\mathbb{Q}_p)w^{-1}, wtw^{-1} \in wT(\mathbb{Q}_p)w^{-1})$ such that:

- $n_{\alpha_0},\ t \in G(\mathcal{O}_F)$,
- $n_{\alpha_0} = 1 \mod p^{m'}$ if $\alpha_0 \in \Phi^-_0$,
- $t = 1 \mod p^{b'}$,
- $n_{\alpha_0} \in U'_{\alpha_0} = \mathrm{Ker}\big(U_{\alpha_0}(\mathbb{Q}_p) \cap G(\mathcal{O}_F) \to \prod_{\alpha \in w^{-1}\Phi^{-,M}, \alpha \mapsto \alpha_0} U_\alpha(\mathcal{O}_F)\big)$.

We let $U''_{\alpha_0} = \mathrm{Im}\big(U'_{\alpha_0} \to \prod_{\alpha\in w^{-1}\Phi_M, \alpha\mapsto\alpha_0} U_\alpha(\mathcal{O}_F)\big)$. We deduce that K_{p,w,M_μ} is in bijection (via the product map) with the elements $((wn_{\alpha_0}w^{-1})_{\alpha_0\in\Phi_0} \in wU''_{\alpha_0}w^{-1}, wtw^{-1} \in wT_{b'}w^{-1})$ such that $n_{\alpha_0} = 1 \mod \varpi^{m'}$ if $\alpha_0 \in \Phi_0^-$. The fact that any element of K_{p,w,M_μ} writes in this way follows from the previous discussion. The injectivity of the product map is a general fact.

We now observe that $w \in {}^MW$ and therefore $\Phi_M^+ \subset w(\Phi^+)$ and $\Phi_M^- \subset w(\Phi^-)$. We deduce that $w^{-1}\Phi_M \cap \Phi^- = w^{-1}\Phi_M^-$ and $w^{-1}\Phi_M \cap \Phi^+ = w^{-1}\Phi_M^+$.

It follows that if $\alpha_0 \in \Phi_0^+$, $wU''_{\alpha_0}w^{-1} \subset U_{M_\mu}(\mathcal{O}_F)$ and if $\alpha_0 \in \Phi_0^-$, $wU''_{\alpha_0}w^{-1} \subset \overline{U}_{M_\mu}(\mathcal{O}_F)$. Therefore, we deduce that $K_{p,w,M_\mu} = N_{p,w,M_\mu} \times wT_{b'}w^{-1} \times \overline{N}_{p,w,M_\mu}$. By the condition that $n_{\alpha_0} = 1 \mod \varpi^{m'}$ if $\alpha_0 \in \Phi_0^-$, we deduce that K_{p,w,M_μ} is a subgroup of the Iwahori of $M_\mu(\mathcal{O}_F)$.

Finally, if G is unramified, and M_μ is defined over $\mathbb{Q}_p$, the partition $\Phi = \Phi_M^+ \coprod \Phi_M^- \coprod \Phi^{+,M} \coprod \Phi^{-,M}$ descends to a partition $\Phi_0 = \Phi_{0,M}^+ \coprod \Phi_{0,M}^- \coprod \Phi_0^{+,M} \coprod \Phi_0^{-,M}$. If w is rational, then it acts on Φ_0. Assume that $K_p = K_{p,1,0}$. The description of K_{p,w,M_μ} simplifies and we find that $U''_{\alpha_0} = \{1\}$ if $\alpha_0 \notin w^{-1}\Phi_{0,M}$, $U''_{\alpha_0} = U_{\alpha_0}(\mathbb{Z}_p)$ if $\alpha_0 \in w^{-1}\Phi_{0,M}$. It follows that K_{p,w,M_μ} is the Iwahori of $M_\mu(\mathbb{Z}_p)$.

□

Example 4.74. The group K_{p,w,M_μ} may be a little strange and need not be open. Let us consider the following example. We assume that $G_{\mathbb{Q}_p}$ is $\mathrm{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\mathrm{GL}_2$, with standard diagonal torus $T_{\mathbb{Q}_p}$ and upper triangular borel $B_{\mathbb{Q}_p}$. We let $K_p = K_{p,1,0}$. We identify $G_{\mathbb{Q}_{p^2}} = \mathrm{GL}_2 \times \mathrm{GL}_2$. We assume that μ is defined over $\mathbb{Q}_{p^2}$ and is given by the cocharacter $t \mapsto \mathrm{diag}(t,1) \times \mathrm{diag}(1,1)$ of $T_{\mathbb{Q}_{p^2}}$. We deduce that P_μ is $B_{\mathbb{Q}_{p^2}} \times \mathrm{GL}_2$, and that M_μ is $T_{\mathbb{Q}_{p^2}} \times GL_2$. We finally observe that $K_p \cap P_\mu$ is $B(\mathbb{Z}_p)$ and therefore $K_{p,1,\mu}$ is the image of $B(\mathbb{Z}_p)$ in $T(\mathbb{Q}_{p^2}) \times GL_2(\mathbb{Q}_{p^2})$.

For all $m, n \in \mathbb{Q}_{\geq 0}$, we let $\mathcal{G}^1_{m,n}$ be the subgroup of $\mathcal{G}$ of elements which reduce to $\mathcal{U}$ modulo $p^{m+\epsilon}$ for all $\epsilon > 0$ and to $\overline{\mathcal{U}}$ modulo p^n (see section 3.3.3). We let $\mathcal{M}^1_{\mu,m,n} \subseteq \mathcal{M}_\mu$ be the group of elements which reduce to $\mathcal{U}_{\mathcal{M}_\mu}$ modulo $p^{m+\epsilon}$ for all $\epsilon > 0$ and to $\overline{\mathcal{U}}_{\mathcal{M}_\mu}$ modulo p^n.

Lemma 4.75. *Let $w \in {}^MW$ and let $K_p = K_{p,m',b'}$ with $m' \in \mathbb{Z}_{>0}$ and $b' \in \mathbb{Z}_{\geq 0}$. Let $m, n \geq 0$ and assume that $0 \leq m - n \leq m' - 1$. Then K_{p,w,M_μ} normalizes $\mathcal{M}^1_{\mu,m,n}$.*

Proof. By lemma 3.17, K_p normalizes $\mathcal{G}^1_{m,n}$. Moreover, $\mathrm{Im}(w\mathcal{G}^1_{m,n}w^{-1} \cap \mathcal{P}_\mu \to \mathcal{M}_\mu) = \mathcal{M}^1_{\mu,m,n}$.

□

It follows from this lemma that $K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n}$ is a subgroup of $\mathcal{M}_\mu$.

Proposition 4.76. *Let $w \in {}^MW$ and let $K_p = K_{p,m',b'}$ with $m' \in \mathbb{Z}_{>0}$ and $m' \geq b'$. Let $m, n \geq 0$ and assume that $0 \leq m-n \leq m'-1$. Over $(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p) \subseteq \mathcal{S}^{tor}_{K^pK_p,\Sigma}$, the torsor $\mathcal{M}_{HT}$ has a reduction to an étale torsor $\mathcal{M}_{HT,m,n,K_p}$ under the group $K^c_{p,w,M_\mu}\mathcal{M}^{1,c}_{\mu,m,n}$.*

Proof. We first do the Hodge case. The proof is very similar to the proof of proposition 4.69. We observe that $]C_{w,k}[_{m,n}K_p = \mathcal{P}^{an}_\mu \backslash \mathcal{P}^{an}_\mu w\mathcal{G}^1_{m,n}K_p$. Since K_p normalizes $\mathcal{G}^1_{m,n}$, $\mathcal{G}^1_{m,n}K_p = K_p\mathcal{G}^1_{m,n}$ is a group. It follows that

$$]C_{w,k}[_{m,n}K_p = (\mathcal{P}^{an}_\mu \cap wK_p\mathcal{G}^1_{m,n}w^{-1})\backslash wK_p\mathcal{G}^1_{m,n}w^{-1}.w$$

We first construct the torsor $\mathcal{M}_{HT,m,n}$ as an open subset of

$$\mathcal{M}_{HT} \times_{S^{tor}_{K,\Sigma}} (\pi^{tor}_{HT})^{-1}(]C_{w,k}[_{m,n}K_p),$$

and prove it is K_p-invariant to descend it to an open subset of

$$\mathcal{M}_{HT} \times_{S^{tor}_{K,\Sigma}} (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p).$$

Let $S \to (\pi^{tor}_{HT})^{-1}(]C_{w,k}[_{m,n}K_p)$ be a map from a perfectoid space S. The torsor $\mathcal{P}_{HT}$ is described as follows: there is a cover $\tilde{S} \to S$ and an element $g_{\tilde{S}} \in \mathcal{G}(\tilde{S})$ such that $h_{\tilde{S}\times_S\tilde{S}} = p_2^\star g_{\tilde{S}}.(p_1^\star g_{\tilde{S}})^{-1} \in \mathcal{P}_\mu(\tilde{S}\times_S \tilde{S})$ is a 1-cocycle describing the torsor. By assumption, we may assume $g_{\tilde{S}} \in w\mathcal{G}^1_{m,n}K_p$, so that $h_{\tilde{S}\times_S\tilde{S}} \in \mathcal{P}_\mu(\tilde{S}\times_S\tilde{S}) \cap w\mathcal{G}^1_{m,n}K_pw^{-1}$ and its image in $\mathcal{M}_\mu$ describes a reduction $\mathcal{M}_{HT,m,n,K_p}$ of $\mathcal{M}_{HT} \times_{S^{tor}_{K,\Sigma}} (\pi^{tor}_{HT})^{-1}(]C_{w,k}[_{m,n}K_p)$ to a $K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n}$-torsor. One checks easily that this torsor is K_p-invariant and therefore descends to a torsor over $(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)$.

The extension to the abelian case proceeds exactly as in the proof of proposition 4.69. We may use section 3.5.2 to make sure that the subset $]C_{w,k}[_{m,n}K_p$ of the flag variety doesn't change when we consider the different groups G_1, B_1, G. □

Proposition 4.77. *Let $w \in {}^MW$ and let $K_p = K_{p,m',b'}$ with $m' \in \mathbb{Z}_{>0}$ and $m' \geq b'$. Let $K'_p = K_{p,m'',b''}$ with $m'' \geq m'$ and $m'' \geq b'' \geq b'$. Let $m, n \geq 0$ and assume that $0 \leq m-n \leq m'-1$. Let $r, s \geq 0$ with $r \geq m$, $s \geq m$ and $0 \leq r-s \leq m''-1$. There is a commutative diagram:*

$$\begin{array}{ccc} \mathcal{M}_{HT,r,s,K'_p} & \longrightarrow & \mathcal{M}_{HT,m,n,K_p} \\ \downarrow & & \downarrow \\ (\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{r,s}K'_p) & \longrightarrow & (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p) \end{array}$$

The top horizontal map is equivariant for the map $K'_{p,w,M_\mu}\mathcal{M}^1_{\mu,r,s} \to K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n}$.

Proof. This follows from the construction of the torsors. □

We have a map $\mathcal{M}^1_{\mu,n,n} \to \mathcal{M}_{\mu,n}$ where $\mathcal{M}_{\mu,n}$ is the sub-group of $\mathcal{M}_\mu$ of elements reducing to 1 modulo p^n. Let $K_p = K_{p,m',b'}$ with $m' > 0$, $m' \geq b'$ and let $m, n \geq 0$ be such that $0 \leq m-n \leq m'-1$. Over $(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$, we define the pushout:

$$\mathcal{M}_{HT,n,K_p} = \mathcal{M}_{HT,n,n,K_p} \times^{K_{p,w,M_\mu}\mathcal{M}^1_{\mu,n,n}} K_{p,w,M_\mu}\mathcal{M}_{\mu,n}.$$

It is sometimes more convenient to work with this torsor because the group $\mathcal{M}_{\mu,n}$ is affinoid.

Proposition 4.78. *Assume that (G, X) is a Hodge-type Shimura datum. Let $K_p = K_{p,m',b}$. For any affinoid open $\mathrm{Spa}(R, R^+) \to (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$ which we assume to be pregood, there exists $K'_p \subseteq K_p$ such that over $\mathrm{Spa}(R,R^+) \times_{\mathcal{S}^{tor}_{K_pK^p,\Sigma}} \mathcal{S}^{tor}_{K'_pK^p,\Sigma}$ the torsor $\mathcal{M}_{HT,n,K_p}$ is trivial.*

Remark 4.79. It will be important for certain vanishing theorems that we are able to prove the triviality of the torsor after a finite flat cover for pregood affinoid opens.

Proof. Let us consider a decreasing sequence of compacts $K_{p,k}$ with $K_{p,0} = K_p$ and $\cap_k K_{p,k} = \{1\}$. Let $\mathrm{Spa}(R_k, R_k^+) = \mathrm{Spa}(R, R^+) \times_{\mathcal{S}^{tor}_{K_pK^p,\Sigma}} \mathcal{S}^{tor}_{K_{p,k}K^p,\Sigma}$. Let $\mathrm{Spa}(R_\infty, R_\infty^+) = \lim \mathrm{Spa}(R_k, R_k^+)$ be an affinoid open of $\mathcal{S}^{tor}_{K^p,\Sigma}$. We first observe that the torsor $\mathcal{M}^{an}_{HT}|_{\mathrm{Spa}(R,R^+)}$ is a Stein space, which can be written as an increasing union of quasi-compact affinoid subsets:

$$\mathcal{M}^{an}_{HT}|_{\mathrm{Spa}(R,R^+)} = \cup_{i\geq 0} (\mathcal{M}^{an}_{HT})_i.$$

Over $\mathrm{Spa}(R_\infty, R_\infty^+)$ we observe that the the torsor $\mathcal{M}_{HT,n,K_p}$ is trivial. Indeed, this torsor is pulled back from the following torsor (over the flag variety):

$$(U_{\mathcal{P}_\mu} \cap wK_p\mathcal{G}_nw^{-1})\backslash wK_p\mathcal{G}_nw^{-1} \to (\mathcal{P}_\mu \cap wK_p\mathcal{G}_nw^{-1})\backslash wK_p\mathcal{G}_nw^{-1}$$

which is trivial because of the Iwahori decomposition of the group $wK_p\mathcal{G}_nw^{-1}$. It follows that $\mathcal{M}_{HT,n,K_p} \times \mathrm{Spa}(R_\infty, R_\infty^+)$ is affinoid, and is a rational open subset of $(\mathcal{M}^{an}_{HT})_i \times \mathrm{Spa}(R_\infty, R_\infty^+)$ for i large enough.

We deduce that $\mathcal{M}_{HT,n,K_p} \times \mathrm{Spa}(R_k, R_k^+)$ is a rational subset of $(\mathcal{M}^{an}_{HT})_i \times \mathrm{Spa}(R_k, R_k^+)$ for k large enough (by approximating the equations defining $\mathcal{M}_{HT,n,K_p} \times \mathrm{Spa}(R_\infty, R_\infty^+)$). Therefore, $\mathcal{M}_{HT,n,K_p} \times \mathrm{Spa}(R_k, R_k^+) = \mathrm{Spa}(T, T^+)$ where (T, T^+) is an (R_k, R_k^+) algebra topologically of finite type. Moreover, there is a section $T^+\hat{\otimes}_{R_k^+} R_\infty^+ \to R_\infty^+$. We now prove that this section can be approximated to a section $T^+\hat{\otimes}_{R_k^+} R_{k'}^+ \to R_{k'}^+$ for k' large enough. By [Elk73], theorem 7, there is a finite type R_k^+-algebra A, such that $A[1/p]$ is smooth over $R_k^+[1/p]$, and whose p-adic completion is isomorphic to T^+. By [Elk73], theorem 2, there exists integers $n_0, r \geq 0$ with the property that for any $n \geq n_0$, for any map of R_k^+-algebras $f : A \to R_{k'}^+/p^n$ there is a map $\tilde{f} : A \to R_{k'}^+$ with the property that $f \mod p^{n-r} = \tilde{f} \mod p^{n-r}$ (the ideal denoted H_B of the reference contains p^h for a large enough integer h, because $A[1/p]$ is smooth over $R_k^+[1/p]$). Let $n = n_0 + r$. We consider the section $s : A \to R_\infty^+$. Its reduction mod p^n factors through $s \mod p^n : A \to R_{k'}^+/p^n$ for k' large enough and therefore we find a lift to a section $\tilde{s} : A \to R_{k'}^+$. □

Remark 4.80. We ask the following question: Let $\mathcal{S}$ be an affinoid adic space and let $\mathcal{H}$ be an affinoid group over $\mathcal{S}$. Let $\mathcal{T} \to \mathcal{S}$ be a $\mathcal{H}$-torsor for the étale topology. Is $\mathcal{T}$ affinoid over $\mathcal{S}$?

4.6.4. *Maps between torsors.* We consider for the moment the following abstract situation. We assume that we have two analytic groups $\mathcal{K} \hookrightarrow \mathcal{G}$. For $i \in \{1, 2\}$, we let $\mathcal{K}_i$ be $\mathcal{K}$-torsors over an adic space $\mathcal{X}$. We let $\mathcal{G}_i$ be their push-out via the map $\mathcal{K} \hookrightarrow \mathcal{G}$. Let $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ be a map of $\mathcal{G}$-torsors over $\mathcal{X}$. Over any cover $\mathcal{U} \to \mathcal{X}$ which trivializes both $\mathcal{K}_1$ and $\mathcal{K}_2$, we can represent the map α by an element $g \in \mathcal{G}(\mathcal{U})$, well defined up to right and left multiplication by $\mathcal{K}(\mathcal{U})$. We shall say that the map α is locally represented (over $\mathcal{U}$) by $\mathcal{K}g\mathcal{K}$. When $\mathcal{K}$ is clear from the context, we also say for simplicity that the map is locally represented by g.

Let (G, X) be a Hodge datum. Let $t \in G(\mathbb{Q}_p)$. Let K_pK^p be a compact open subgroup. For suitable choices of polyhedral cone decomposition, we have a correspondence:

$$\begin{array}{ccc} & S^{tor}_{K^p(K_p\cap tK_pt^{-1}),\Sigma''} & \\ \swarrow^{p_2} & & \searrow^{p_1} \\ S^{tor}_{K^pK_p,\Sigma} & & S^{tor}_{K^pK_p,\Sigma'} \end{array}$$

We get an associated maps of pro-Kummer-étale right torsors:

$$\begin{array}{ccc} p_2^\star\mathcal{G}^{an}_{pet} & \longrightarrow & p_1^\star\mathcal{G}^{an}_{pet} \\ \uparrow & & \uparrow \\ p_2^\star\mathcal{G}_{pet} & & p_1^\star\mathcal{G}_{pet} \end{array}$$

which by definition is locally represented by K_ptK_p. We also deduce a map of étale right torsors:

$$\begin{array}{ccc} p_2^\star\mathcal{M}^{an}_{HT} & \longrightarrow & p_1^\star\mathcal{M}^{an}_{HT} \\ \uparrow & & \uparrow \\ p_2^\star\mathcal{M}_{HT} & & p_1^\star\mathcal{M}_{HT} \end{array}$$

Let $w \in {}^MW$ and let $K_p = K_{p,m',b'}$ with $m' > 0$ and $b' \geq 0$. Let $m,n \geq 0$ and assume that $0 \leq m-n \leq m'-1$. Over $p_2^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)\big) \cap p_1^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)\big)$, we have a map of étale right torsors:

$$\begin{array}{ccc} p_2^\star\mathcal{M}^{an}_{HT} & \longrightarrow & p_1^\star\mathcal{M}^{an}_{HT} \\ \uparrow & & \uparrow \\ p_2^\star\mathcal{M}_{HT} & & p_1^\star\mathcal{M}_{HT} \\ \uparrow & & \uparrow \\ p_2^\star\mathcal{M}_{HT,m,n,K_p} & & p_1^\star\mathcal{M}_{HT,m,n,K_p} \end{array}$$

Proposition 4.81. *Let $w \in {}^MW$.*

Let $t \in T(\mathbb{Q}_p)$. The map $p_2^\star\mathcal{M}^{an}_{HT} \to p_1^\star\mathcal{M}^{an}_{HT}$ restricted to

$$p_2^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)\big) \cap p_1^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)\big)$$

is locally represented by

$$K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n}wtw^{-1}K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n}.$$

Proof. Locally for the pro-Kummer-étale topology we have sections $x_2 \in p_2^\star\mathcal{G}_{pet}$ and $x_1 \in p_1^\star\mathcal{G}_{pet}$. Note that x_1 and x_2 are well defined up to translation by K_p. There is an isomorphism:

$$\begin{array}{ccc} p_2^\star\mathcal{G}^{an}_{pet} & \longrightarrow & p_1^\star\mathcal{G}^{an}_{pet} \\ \uparrow & & \uparrow \\ x_2G(\mathbb{Q}_p) & \xrightarrow{t} & x_1G(\mathbb{Q}_p) \end{array}$$

where the bottom map is $x_2g \mapsto x_1tg$. Therefore, the map $p_2^\star\mathcal{G}_{pet}^{an} \to p_1^\star\mathcal{G}_{pet}^{an}$ is locally represented by t. We now get by pushforward to $\mathcal{G}^{an}$ a diagram:

$$\begin{array}{ccc} p_2^\star\mathcal{G}_{pet}^{an} \times_{G(\mathbb{Q}_p)} \mathcal{G}^{an} & \longrightarrow & p_1^\star\mathcal{G}_{pet}^{an} \times_{G(\mathbb{Q}_p)} \mathcal{G}^{an} \\ \uparrow & & \uparrow \\ x_2\mathcal{G}^{an} & \xrightarrow{\quad t \quad} & x_1\mathcal{G}^{an} \end{array}$$

The torsors $p_2^\star\mathcal{G}_{pet}^{an}$ and $p_1^\star\mathcal{G}_{pet}^{an}$ arise by pushforward from torsors $p_1^\star\mathcal{P}_{HT}^{an}$ and $p_2^\star\mathcal{P}_{HT}^{an}$ and we can locally find a diagram for trivializations x_2' and x_1':

$$\begin{array}{ccc} p_2^\star\mathcal{P}_{HT}^{an} & \longrightarrow & p_1^\star\mathcal{P}_{HT}^{an} \\ \uparrow & & \uparrow \\ x_2'\mathcal{P}_{\mu}^{an} & \longrightarrow & x_1'\mathcal{P}_{\mu}^{an} \end{array}$$

We pick $w \in {}^MW$ and we work locally over

$p_2^{-1}\big((\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{m,n}K_p)\big) \cap p_1^{-1}\big((\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{m,n}K_p)\big) \subseteq S^{tor}_{K^p(K_p\cap tK_pt^{-1}),\Sigma''}$.

Concretely, this means that $x_2 = x_2'wh_2$ and $x_1 = x_1'wh_1$ for $h_i \in \mathcal{G}^1_{m,n}K_p$ and the bottom map is described by

$$x_2'p = x_2h_2^{-1}w^{-1}p \mapsto x_1th_2^{-1}w^{-1}p = x_1'wh_1th_2^{-1}w^{-1}p.$$

This forces $wh_1th_2^{-1}w^{-1} \in \mathcal{P}_\mu^{an}$.

By pushforward to $\mathcal{M}_{HT}^{an}$ we get:

$$\begin{array}{ccc} p_2^\star\mathcal{M}_{HT}^{an} & \longrightarrow & p_1^\star\mathcal{M}_{HT}^{an} \\ \uparrow & & \uparrow \\ x_2'\mathcal{M}_{\mu}^{an} & \longrightarrow & x_1'\mathcal{M}_{\mu}^{an} \end{array}$$

where x_2' and x_1' are now viewed as sections of $p_2^\star\mathcal{M}_{HT,m,n,K_p}$ and $p_1^\star\mathcal{M}_{HT,m,n,K_p}$ and the bottom map is

$$x_2'm \mapsto x_1'\overline{(wh_1th_2^{-1}w^{-1})}m$$

where $\overline{(wh_1th_2^{-1}w^{-1})}$ is the image of $(wh_1th_2^{-1}w^{-1})$ via the map $\mathcal{P}_\mu^{an} \to \mathcal{M}_\mu^{an}$. By lemma 4.82, $(wh_1th_2^{-1}w^{-1}) \in K_{p,w,M_\mu}\mathcal{M}_nwtw^{-1}K_{p,w,M_\mu}\mathcal{M}^1_{m,n}$.

□

Lemma 4.82. *For any $t \in T(\mathbb{Q}_p)$, we have that*

$$\mathrm{Im}(wK_p\mathcal{G}^1_{m,n}tK_p\mathcal{G}^1_{m,n}w^{-1} \cap \mathcal{P}_\mu^{an} \to \mathcal{M}_\mu^{an}) = K_{p,w,M_\mu}\mathcal{M}^1_{m,n}wtw^{-1}K_{p,w,M_\mu}\mathcal{M}^1_{m,n}.$$

Proof. We will prove that $(wK_p\mathcal{G}^1_{m,n}tK_p\mathcal{G}^1_{m,n}w^{-1}) \cap \mathcal{P}_\mu^{an} =$

$$((wK_p\mathcal{G}^1_{m,n}w^{-1}) \cap \mathcal{P}_\mu^{an})wtw^{-1}((wK_p\mathcal{G}^1_{m,n}w^{-1}) \cap \mathcal{P}_\mu^{an}).$$

Any element in $k \in wK_p\mathcal{G}^1_{m,n}w^{-1}$ writes uniquely $\prod_{\alpha\in\Phi} k_\alpha$ (for any fixed ordering of the roots). Let

$$kwtw^{-1}k' \in (wK_p\mathcal{G}^1_{m,n}w^{-1}wtw^{-1}wK_p\mathcal{G}^1_{m,n}w^{-1}) \cap \mathcal{P}_\mu^{an}$$

with $k = \prod_{\alpha \in \Phi^+ \cup \Phi^-_M} k_\alpha \prod_{\alpha \in \Phi^{-,M}} k_\alpha$ and $k' = \prod_{\alpha \in \Phi^{-,M}} k'_\alpha \prod_{\alpha \in \Phi^+ \cup \Phi^-_M} k'_\alpha$. A necessary and sufficient condition that $kwtw^{-1}k' \in \mathcal{P}^{an}_\mu$ is that $k''wtw^{-1}k''' \in \mathcal{P}^{an}_\mu$ where $k'' = \prod_{\alpha \in \Phi^-_M} k_\alpha$ and $k''' = \prod_{\alpha \in \Phi^-_M} k'''_\alpha$. But $k''wtw^{-1}k''' \in \mathcal{U}_{\overline{P}^{an}_\mu} \rtimes \mathcal{T}^{an}$ and necessarily, $k'' = k''' = 1$. $\square$

We now check that proposition 4.81 continues to hold in the abelian case.

Proposition 4.83. *Let (G, X) be an abelian datum. Let $w \in {}^MW$. Let $t \in T(\mathbb{Q}_p)$. The map $p_2^\star \mathcal{M}^{an}_{HT} \to p_1^\star \mathcal{M}^{an}_{HT}$ restricted to*

$$p_2^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)\big) \cap p_1^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)\big)$$

is locally represented by

$$K^c_{p,w,M_\mu}\mathcal{M}^{1,c}_{\mu,m,n} wtw^{-1} K^c_{p,w,M_\mu}\mathcal{M}^{1,c}_{\mu,m,n}.$$

Proof. It suffices to prove the statement over $\mathcal{S}^{an}_{K_pK^p}$ by lemma 4.55. Over $\mathcal{S}^{an}_{K_pK^p}$, we can reproduce the argument given in the Hodge case. $\square$

5. Overconvergent cohomologies and the spectral sequence

Our goal in this section is to introduce a spectral sequence which computes classical finite slope cohomology in terms of the finite slope parts of certain overconvergent cohomologies indexed by $w \in {}^MW$. Moreover we will prove a classicality theorem comparing the small slope part of classical cohomology in regular weight with the small slope part of a single overconvergent cohomology for a w determined by the weight. We will also prove for possibly non regular weights a vanishing theorem for the classical cohomology.

5.1. Correspondences and cohomology with support.

We begin by discussing the action of a cohomological correspondence on the cohomology with support of a sheaf. Let $(F, \mathcal{O}_F)$ be a non-archimedean local field and let $\mathcal{X}$ be an adic space locally of finite type over $\mathrm{Spa}(F, \mathcal{O}_F)$. Let

$$\begin{array}{ccc} & \mathcal{C} & \\ {}^{p_2}\swarrow & & \searrow^{p_1} \\ \mathcal{X} & & \mathcal{X} \end{array}$$

be a correspondence, where $\mathcal{C}$ is locally of finite type and p_1 and p_2 are morphisms of adic spaces.

5.1.1. *Action of the correspondences on subsets of $\mathcal{X}$.* Let $\mathcal{P}(\mathcal{X})$ be the set of subsets of $\mathcal{X}$. Let us denote by $T : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ the map which takes $\mathcal{B} \in \mathcal{P}(\mathcal{X})$ to $p_2(p_1^{-1})(\mathcal{B})$ and $T^t : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ the map which takes $\mathcal{B} \in \mathcal{P}(\mathcal{X})$ to $p_1(p_2^{-1})(\mathcal{B})$.

Lemma 5.1. *Assume that p_1 is finite flat (see* [Hub96]*, section 1.3 and 1.4). The map T^t sends closed subsets to closed subsets and open subsets to open subsets.*

Proof. Let $\mathcal{Z}$ be a closed subset of $\mathcal{X}$. Then $p_2^{-1}\mathcal{Z}$ is a closed subset of $\mathcal{C}$ and $T^t(\mathcal{Z})$ is closed by properness. Let $\mathcal{U} \subseteq \mathcal{X}$ be an open subset. Then $p_2^{-1}(\mathcal{U})$ is open in $\mathcal{C}$ and $T^t(\mathcal{U})$ is open since finite flat morphisms are open ([Hub96], lemma 1.7.9). $\square$

5.1.2. *Action of the correspondence on a sheaf.* Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_{\mathcal{X}}$-modules. Let $\mathcal{U} \subseteq \mathcal{X}$ be an open subset of $\mathcal{X}$ and let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed subset. We assume that $T(\mathcal{U})$ is open and that p_1 is finite flat. We also assume that we have a map $T : p_2^\star \mathscr{F} \to p_1^\star \mathscr{F}$. We can define a map

$$T : \mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}), \mathscr{F}) \to \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap \mathcal{U}}(\mathcal{U}, \mathscr{F})$$

as the following composite:

$$\mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}), \mathscr{F}) \stackrel{a}{\to} \mathrm{R}\Gamma_{p_2^{-1}(\mathcal{Z}\cap T(\mathcal{U}))}(p_2^{-1}(T(\mathcal{U})), p_2^\star\mathscr{F}) \stackrel{b}{\to} \mathrm{R}\Gamma_{p_2^{-1}(\mathcal{Z})\cap p_1^{-1}(\mathcal{U})}(p_1^{-1}(\mathcal{U}), p_2^\star\mathscr{F})$$

$$\stackrel{c}{\to} \mathrm{R}\Gamma_{p_2^{-1}(\mathcal{Z})\cap p_1^{-1}(\mathcal{U})}(p_1^{-1}(\mathcal{U}), p_1^\star\mathscr{F}) \stackrel{d}{\to} \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap \mathcal{U}}(\mathcal{U}, \mathscr{F}),$$

where:

- a is a pull back map along $p_2^{-1}(T(\mathcal{U})) \to T(\mathcal{U})$,
- b is a pull back map along $p_1^{-1}(\mathcal{U}) \to p_2^{-1}(T(\mathcal{U}))$. Notice that $p_2^{-1}(\mathcal{Z} \cap T(\mathcal{U})) \cap p_1^{-1}(\mathcal{U}) = p_2^{-1}(\mathcal{Z}) \cap p_1^{-1}(\mathcal{U})$,
- c is given by the map $T : p_2^\star\mathscr{F} \to p_1^\star\mathscr{F}$,
- d is given by the trace map of lemma 2.2. Notice that $p_1(p_2^{-1}(\mathcal{Z}) \cap p_1^{-1}(\mathcal{U})) \subset T^t(\mathcal{Z}) \cap \mathcal{U}$.

Remark 5.2. We observe that in the definition of the correspondence, the map $p_2^\star\mathscr{F} \to p_1^\star\mathscr{F}$ is only used in a neighborhood of

$$p_2^{-1}(\mathcal{Z}) \cap p_1^{-1}(\mathcal{U}).$$

Indeed, let $\mathcal{V}$ be a neighborhood of $p_2^{-1}(\mathcal{Z}) \cap p_1^{-1}(\mathcal{U})$ in $\mathcal{C}$, then we have

$$\mathrm{R}\Gamma_{p_2^{-1}(\mathcal{Z})\cap p_1^{-1}(\mathcal{U})}(p_1^{-1}(\mathcal{U}), p_i^\star\mathscr{F}) = \mathrm{R}\Gamma_{p_2^{-1}(\mathcal{Z})\cap p_1^{-1}(\mathcal{U})}(\mathcal{V}, p_i^\star\mathscr{F})$$

for $i = 1, 2$.

5.2. **The finite slope part.** We briefly recall the spectral theory of compact operators over a non-archimedean field.

5.2.1. *Slope decomposition.* Let F be a non archimedean field extension of $\mathbb{Q}_p$. The valuation v on F is normalized by $v(p) = 1$. Let M be a vector space over F and let T be an endomorphism of the vector space M. Let $h \in \mathbb{Q}$. A h-slope decomposition of M with respect to T is a direct sum decomposition of F-vector spaces $M = M^{\leq h} \oplus M^{>h}$ such that:

(1) $M^{\leq h}$ and $M^{>h}$ are stable under the action of T.
(2) $M^{\leq h}$ is finite dimensional over F.
(3) All the eigenvalues (in an algebraic closure of F) of T acting on $M^{\leq h}$ are of valuation less or equal to h.
(4) For any polynomial Q with roots of valuation strictly greater than h, the restriction of $Q^*(T)$ to $M^{>h}$ is an invertible endomorphism. Here Q^* is the reciprocal of Q.

By [Urb11], coro. 2.3.3, if such a slope decomposition exists, it is unique. If M has h-slope decomposition for all $h \in \mathbb{Q}$, we simply say that M has slope decomposition. In this situation we can obviously define submodules $M^{=h}$ and $M^{<h}$ of M for all $h \in \mathbb{Q}$. We let $M^{fs} = \cup_h M^{\leq h}$ be the union of all the slope $\leq h$ factors of M and we call it the finite slope part of M with respect to T.

5.2.2. *Compact operators.* Let F be a non archimedean field extension of $\mathbb{Q}_p$. Let $M \in \mathbf{Ban}(F)$ and let T be a compact endomorphism of K. Then by [Ser62], M has a slope decomposition with respect to T. More generally, let $M^\bullet \in Ob(\mathcal{K}^{proj}(F))$ and let $T \in \mathrm{End}_{\mathcal{D}(\mathbf{Ban}(F))}(M^\bullet)$ be a compact operator. By fixing a compact representative $\tilde{T}$ of T, we can define, for all $h \in \mathbb{Q}$, a direct factor $M^{\bullet, \leq h} \in Ob(\mathcal{K}^{perf}(F))$ of $M^\bullet$, called the slope less than h part of $M^\bullet$. This does not depend on the choice of $\tilde{T}$. We let $M^{\bullet, fs} = \mathrm{colim}_{h \geq 0} M^{\bullet, \leq h}$ be the finite slope part of $M^\bullet$.

5.2.3. *Action of an algebra.* Let T^+ be a monoid (with neutral element) and $T^{++} \subseteq T^+$ be a sub-monoid (possibly without neutral element), such that $T^+.T^{++} \subseteq T^{++}$. We also assume that for any $t, t' \in T^{++}$, there exists $n \in \mathbb{N}$, $t'' \in T^+$ such that $t^n = t't''$. Let $M^\bullet \in Ob(\mathcal{K}^{proj}(F))$. We assume that we have an algebra action $\mathbb{Z}[T^+] \hookrightarrow \mathrm{End}(M^\bullet)$, such that the ideal $\mathbb{Z}[T^{++}]$ acts by potent compact operators. For this later property to hold, it actually suffices that a single element $t \in T^{++}$ acts compactly.

Lemma 5.3. *For any $t, t' \in T^{++}$ acting compactly on $M^\bullet$, the corresponding finite slope parts $M^{\bullet, t-fs}$ and $M^{\bullet, t'-fs}$ are canonically quasi-isomorphic.*

Proof. We reduce to the case that $M^\bullet$ is an object M of $\mathbf{Ban}(F)$. We have $M = M^{t-fs} \oplus M^{t-\infty s}$ and $M = M^{t'-fs} \oplus M^{t'-\infty s}$ the decompositions of M into the direct sum of the finite slope part where t (resp. t') is invertible and the infinite slope part where t (resp. t') is topologically nilpotent. There exists $n \in \mathbb{N}$, $t'' \in T^+$ such that $t^n = t't''$. We deduce that $M^{t'-fs} \subseteq M^{t-fs}$ and $M^{t-\infty s} \subseteq M^{t'-\infty s}$. Exchanging the roles of t and t' we deduce the lemma. □

In view of this lemma, we use the notation $M^{\bullet, fs}$ to mean $M^{\bullet, t-fs}$ for any compact operator $t \in T^{++}$.

5.2.4. *Projective limits of Banach spaces.* We briefly explain how the theory extends to the case where the complex $M^\bullet \in Ob(\mathcal{K}^{proj}(F))$ is replaced by an object "$\lim_i$"$M_i^\bullet \in Ob(\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F)))$. Therefore, we let T be a compact operator of "$\lim_i$"$M_i^\bullet \in Ob(\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F)))$. By lemma 2.6, T induces canonically a compact endomorphism T_i of $M_i^\bullet$ for i large enough and there are factorization diagrams:

$$\begin{array}{ccc} M_{i+1}^\bullet & \xrightarrow{T_{i+1}} & M_{i+1}^\bullet \\ \downarrow & \nearrow & \downarrow \\ M_i^\bullet & \xrightarrow{T_i} & M_i^\bullet \end{array}$$

For any $h \in \mathbb{Q}$, we deduce that $M_{i+1}^{\bullet, \leq h} \to M_i^{\bullet, \leq h}$ is a quasi-isomorphism. We can therefore define ("$\lim_i$"$M_i^\bullet)^{\leq h} = \lim_i M_i^{\bullet, \leq h} \in Ob(\mathcal{K}^{perf}(F))$ and ("$\lim_i$"$M_i^\bullet)^{fs} = \mathrm{colim}_h(($"$\lim_i$"$M_i^\bullet)^{\leq h})$.

Granting these facts, all the material developed in section 5.2.1 and section 5.2.3 applies in the more general setting of objects of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$.

5.3. **A formal analytic continuation result.** In this section we prove a result which will identify the finite slope part of different cohomologies. This can be seen as an (abstract) generalization of [Buz03], thm. 5.2. We recall the setting. Let $\mathcal{X}$ be an adic space locally of finite type over $\mathrm{Spa}(F, \mathcal{O}_F)$, and $p_1, p_2 : \mathcal{C} \to \mathcal{X}$ is a correspondence. We assume that p_1 is finite flat. We also let $\mathscr{F}$ be a projective

Banach sheaf (see definition 2.7) and we assume that the map $T : p_2^\star \mathscr{F} \to p_1^\star \mathscr{F}$ is compact (see definition 2.8). In this part 5 of the article, we will only consider the case where $\mathscr{F}$ is a projective coherent sheaf. In this particular setting, any map $p_2^\star \mathscr{F} \to p_1^\star \mathscr{F}$ is compact.

5.3.1. *The diamond of a correspondence.* We now make the further assumption that $T^t(\mathcal{Z}) \subseteq \mathcal{Z}$ and $T(\mathcal{U}) \subseteq \mathcal{U}$. We can build the following diamond:

$$
\begin{array}{ccccc}
 & & \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) & & \\
 & \swarrow^{res} & & \nwarrow^{cores} & \\
\mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F}) & & \xrightarrow{\quad T \quad} & & \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \\
 & \nwarrow_{cores} & & \swarrow_{res} & \\
 & & \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F}) & &
\end{array}
$$

and the composition of the top and lower triangle $res \circ cores$ and $cores \circ res$ are equal. We can define an endomorphism of $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F})$, $\mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F})$, $\mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap\mathcal{U}}(\mathcal{U},\mathscr{F})$ and $\mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F})$ by composing T with the restriction and corestriction maps. We call this endomorphism T by abuse of notation.

Proposition 5.4. *Assume that all the complexes appearing are objects of* $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$ *and that* T *is potent compact. After applying the finite slope projector, all the maps in the resulting diamond*

$$
\begin{array}{ccccc}
 & & \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F})^{fs} & & \\
 & \swarrow^{res} & & \nwarrow^{cores} & \\
\mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F})^{fs} & & \xrightarrow{\quad T \quad} & & \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap\mathcal{U}}(\mathcal{U},\mathscr{F})fs \\
 & \nwarrow_{cores} & & \swarrow_{res} & \\
 & & \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F})^{fs} & &
\end{array}
$$

are quasi-isomorphisms.

Proof. This follows from the following elementary result. Let $\{M_i\}_{i\in\mathbb{Z}/n\mathbb{Z}}$ be abelian groups and let $f_i : M_i \to M_{i+1}$ be maps. If for all $i \in \mathbb{Z}/n\mathbb{Z}$, the map $f_{i-1} \circ f_{i-2} \cdots \circ f_i : M_i \to M_i$ is an isomorphism, then all the maps f_i are isomorphisms. □

Remark 5.5. The operator T is potent compact if the map "restriction-corestriction" obtained by composing the top or low arrows of the diamond $\mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F})$ is compact (see lemma 2.24 for a criterium).

As a corollary, we note the following fact:

Corollary 5.6. *Under the hypothesis of proposition 5.4, let* $\mathcal{U}'$ *and* $\mathcal{Z}'$ *be open and closed subsets, such that* $T(\mathcal{U}) \cap \mathcal{Z} \subseteq \mathcal{U}' \subseteq \mathcal{U}$ *and* $\mathcal{U} \cap T^t(\mathcal{Z}) \subseteq \mathcal{Z}' \subseteq \mathcal{Z}$. *Assume also that* $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U},\mathscr{F})$ *is an object of* $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$. *Then the operator* T *is well defined and potent compact on* $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F})$ *and the finite slope part of* $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F})$ *is canonically quasi-isomorphic to the finite slope part of* $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F})$.

Proof. Under our assumptions, we have restriction-corestriction maps:

$$\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}') \to \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'\cap T(\mathcal{U})}(\mathcal{U}'\cap T(\mathcal{U})) \to \mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U})\cap\mathcal{U}',\mathscr{F}) = \mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F})$$

and

$$\mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \to \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F}).$$

We can build the following diagram:

$$\begin{array}{ccccc} & & \mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F}) & & \\ & \swarrow & & \nwarrow & \\ \mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F}) & & \xrightarrow{\quad T \quad} & & \mathrm{R}\Gamma_{T^t(\mathcal{Z})\cap\mathcal{U}}(\mathcal{U},\mathscr{F}) \end{array}$$

and T is therefore well defined and potent compact on $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F})$. We deduce that $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}',\mathscr{F})^{fs}$ is quasi-isomorphic to $\mathrm{R}\Gamma_{\mathcal{Z}\cap T(\mathcal{U})}(T(\mathcal{U}),\mathscr{F})^{fs}$ which is in turn quasi-isomorphic to $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U},\mathscr{F})^{fs}$. □

5.3.2. *The infinite diamond.* It is interesting to iterate the operator T. We now work under the stronger assumption that p_1 and p_2 are finite flat. These assumptions imply that for any $n \geq 0$, the n-th iterate $\mathcal{C}^{(n)}$ of the correspondence $\mathcal{C}$ comes with two finite flat projections $p_{1,n}$ and $p_{2,n}$.

Remark 5.7. Later we will apply this material to toroidal compactifications of Shimura varieties. We therefore have to work under slightly more general assumptions. Namely, it is not possible in general to find cone decompositions such that all the maps between compactifications of Hecke correspondences are finite flat. Nevertheless, by allowing suitable changes of the cone decompositions, we can always assume that a given map is finite flat. Moreover, the composition of compactified Hecke correspondences and their action on the cohomology has been explained in detail in section 4.2.2. Therefore, for the clarity of the exposition, we will keep the assumption that p_1 and p_2 are finite flat here.

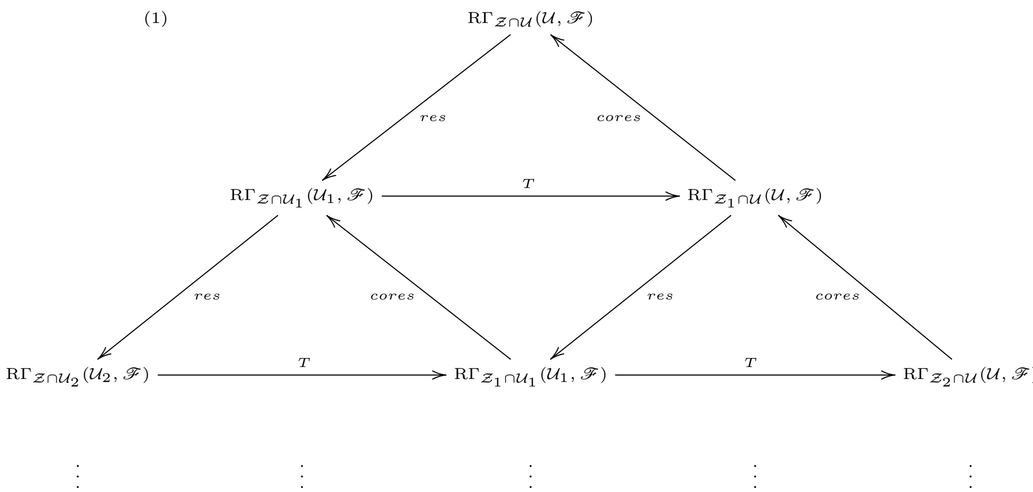
We let $\mathcal{U}_m = T^m(\mathcal{U})$ and $\mathcal{Z}_n = (T^t)^n(\mathcal{Z})$. We assume that $T(\mathcal{U}) \subseteq \mathcal{U}$ and $T^t(\mathcal{Z}) \subseteq \mathcal{Z}$. The sequences $\{\mathcal{U}_m\}_{m\geq 0}$ and $\{\mathcal{Z}_n\}_{n\geq 0}$ are therefore decreasing.

We can then construct diamonds as above for all $n, m \geq 0$:

$$\begin{array}{ccccc} & & \mathrm{R}\Gamma_{\mathcal{Z}_n\cap\mathcal{U}_m}(\mathcal{U}_m,\mathscr{F}) & & \\ & \swarrow^{res} & & \nwarrow^{cores} & \\ \mathrm{R}\Gamma_{\mathcal{Z}_n\cap\mathcal{U}_{m+1}}(\mathcal{U}_{m+1},\mathscr{F}) & & \xrightarrow{\quad T \quad} & & \mathrm{R}\Gamma_{\mathcal{Z}_{n+1}\cap\mathcal{U}_m}(\mathcal{U}_m,\mathscr{F}) \\ & \nwarrow_{cores} & & \swarrow_{res} & \\ & & \mathrm{R}\Gamma_{\mathcal{Z}_{n+1}\cap\mathcal{U}_{m+1}}(\mathcal{U}_{m+1},\mathscr{F}) & & \end{array}$$

and we can add them to get an infinite diamond diagram looking like:

(1)



We assume that all the objects of the above diagram belong to $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$. We now make the further assumption that there exists (m_0, n_0) such that one morphism "restriction-corestriction" obtained by composing the top or low arrows of a diamond $\mathrm{R}\Gamma_{\mathcal{Z}_{n_0+1}\cap\mathcal{U}_{m_0}}(\mathcal{U}_{m_0}, \mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}_{n_0}\cap\mathcal{U}_{m_0+1}}(\mathcal{U}_{m_0+1}, \mathscr{F})$ is compact.

For any m, n with $m, n \geq 0$ and $(m, n) \neq (0, 0)$, we can define an endomorphism $T_{m,n} : \mathrm{R}\Gamma_{\mathcal{Z}_n\cap\mathcal{U}_m}(\mathcal{U}_m, \mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{Z}_n\cap\mathcal{U}_m}(\mathcal{U}_m, \mathscr{F})$ by composing T, *res* and *cores* in a suitable order. We abuse notation and denote this operator by T. The operator T is potent compact because some power of it will factor over the compact "restriction-corestriction" map above. In any case, we can speak of the finite slope direct factor of $\mathrm{R}\Gamma_{\mathcal{Z}_n\cap\mathcal{U}_m}(\mathcal{U}_m, \mathscr{F})$ for T.

Theorem 5.8. *On the finite slope part, all the morphisms of the infinite diamond are quasi-isomorphisms.*

Proof. This follows from proposition 5.4. □

Corollary 5.9. *Under the hypothesis of theorem 5.8, assume that there is $m, n, s \in \mathbb{Z}_{\geq 0}$ such that $(T^t)^{n+s}(\mathcal{Z}) \cap T^m(\mathcal{U}) \subseteq \mathcal{Z}' \subset (T^t)^n(\mathcal{Z})$ is a closed subset and $T^{m+s}(\mathcal{U}) \cap (T^t)^n(\mathcal{Z}) \subset \mathcal{U}' \subset T^m(\mathcal{U})$. Assume moreover that $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}', \mathscr{F})$ is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$. Then the operator T^s is well defined and potent compact on $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}', \mathscr{F})$ and the finite slope part of $\mathrm{R}\Gamma_{\mathcal{Z}'\cap\mathcal{U}'}(\mathcal{U}', \mathscr{F})$ is canonically quasi-isomorphic to the finite slope part of $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathscr{F})$.*

Proof. This follows from corollary 5.6. □

5.4. **Overconvergent cohomologies.** Let (G, X) be an abelian datum such that $G_{\mathbb{Q}_p}$ is quasi split. Let $w \in {}^MW$. For a choice of $+$ or $-$ and a weight $\kappa \in X^\star(T^c)^{M_\mu,+}$ we want to define a finite slope overconvergent cohomology $\mathrm{R}\Gamma_w(K^p, \kappa)^{\pm,fs}$ and the cuspidal counterpart $\mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{\pm,fs}$ by taking cohomologies with suitable support conditions of neighborhoods of the inverse image of $\mathcal{P}_\mu\backslash\mathcal{P}_\mu w K_p$, $w \in {}^MW$ by the Hodge-Tate period map, and applying a finite slope projector. We will also define variants $\mathrm{R}\Gamma_w(K^p, \kappa, \chi)^{\pm,fs}$ and $\mathrm{R}\Gamma_w(K^p, \kappa, \chi, cusp)^{\pm,fs}$ where $\chi : T(\mathbb{Z}_p) \to F^\times$ is a finite order character.

5.4.1. *First definition.* For a level $K_p = K_{p,m',b}$ with $m' \geq b \geq 0$ and $m' > 0$ and a weight $\kappa \in X^\star(T^c)^{M_\mu,+}$, we define:

$$\mathrm{R}\Gamma_w(K^pK_p,\kappa)^{+,fs} := \mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{0,\overline{0}})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[),\mathcal{V}_\kappa)^{+,fs}.$$

Implicit in this definition is that this cohomology is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$ and that $\mathcal{H}^+_{p,m',b}$ acts on it in a way that $\mathcal{H}^{++}_{p,m',b}$ acts by potent compact operators (this will be proved below in Theorem 5.10).

Similarly for a weight $\kappa \in X^\star(T^c)^{M_\mu,+}$, we define:

$$\mathrm{R}\Gamma_w(K^pK_p,\kappa)^{-,fs} := \mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{\overline{0},0})}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[),\mathcal{V}_\kappa)^{-,fs}.$$

Again implicit in this definition is that this cohomology is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$ and that $\mathcal{H}^-_{p,m',b}$ acts on it in a way that $\mathcal{H}^{--}_{p,m',b}$ acts by potent compact operators (this will also be proved below in Theorem 5.10).

We have similar definitions for cuspidal cohomology.

5.4.2. *Existence of finite slope cohomology.*

Theorem 5.10. *Let $K_p = K_{p,m',b}$ for some $m' \geq b \geq 0$ with $m' > 0$, and fix $w \in {}^MW$ and $\kappa \in X^\star(T^c)^{M_\mu,+}$.*

(1) *There is an action of $\mathcal{H}^+_{p,m',b}$ on $\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{0,\overline{0}})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[),\mathcal{V}_\kappa)$ for which $\mathcal{H}^{++}_{p,m',b}$ acts via compact operators. The same statement holds for $\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{0,\overline{0}})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[),\mathcal{V}_\kappa(-D))$.*

(2) *There is an action of $\mathcal{H}^-_{p,m',b}$ on $\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{\overline{0},0})}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[),\mathcal{V}_\kappa)$ for which $\mathcal{H}^{--}_{p,m',b}$ acts via compact operators. The same statement holds for $\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{\overline{0},0})}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[),\mathcal{V}_\kappa(-D))$.*

Proof. We only prove point 1 for non cuspidal cohomology. The rest of the argument is very similar and left to the reader.

Let $U = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]X_{w,k}[)$ and $Z = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(\overline{]Y_{w,k}[})$. By lemma 3.21 we have

$$\mathrm{R}\Gamma_{U\cap Z}(U,\mathcal{V}_\kappa) = \mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{0,\overline{0}})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[),\mathcal{V}_\kappa).$$

Now if T is the Hecke operator $[K_{p,m',b}tK_{p,m',b}]$ for any $t \in T^+$, we have $T(U) \subseteq U$ and $T^t(Z) \subseteq Z$ by lemma 3.29 and hence there is an action of $\mathcal{H}^+_{p,m',b}$ on $\mathrm{R}\Gamma_{U\cap Z}(U,\mathcal{V}_\kappa)$ via the construction explained at the beginning of Section 5.3. That this action defines an action of the Hecke algebra $\mathcal{H}^+_{p,m',b}$ follows from the discussion of section 4.2.2.

Now suppose that T is associated to $t \in T^{++}$. In order to simplify notations, we choose t such that $\min(t) = \inf_{\alpha\in\Phi^+} v(\alpha(t)) \geq 1$. In order to show the action of T is potent compact, it suffices to show that the "restriction-corestriction" map

$$\mathrm{R}\Gamma_{U\cap T^t(Z)}(U,\mathcal{V}_\kappa) \to \mathrm{R}\Gamma_{T(U)\cap Z}(T(U),\mathcal{V}_\kappa)$$

is compact. We need to check the assumptions of lemma 2.24.

We will be done if we can find a quasi-compact open subset U' such that $T(U)\cap Z \subseteq U'$ and $\overline{U'} \subseteq U$, and closed subset Z_1, Z_2, with quasi-compact complement, such that $U \cap T^t(Z) \subseteq Z_1$, $Z_1 \subseteq \overset{\circ}{Z}_2$ and $Z_2 \subseteq Z$. By lemma 3.29, we have:

$$U \cap T^t(Z) \subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}\big(]C_{w,k}[_{0,\overline{1}}K_{p,m',b}\big)$$

and

$$T(U) \cap Z \subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}\big(]C_{w,k}[_{1,\overline{0}}K_{p,m',b}\big)$$

We first find U. By lemma 3.21,

$$(\pi^{tor}_{HT,K_{p,m',b}})^{-1}\big(]C_{w,k}[_{0,-\infty}K_{p,m',b}\big) \subseteq U.$$

It follows that $\overline{T(U) \cap Z} \subseteq U$. We now observe that $]X_{w,k}[= \cup_n]X_{w,k}[_n$ where $]X_{w,k}[_n$ is the (quasi-compact) tube of radius $|p^{1/n}|$. This is a covering of $]X_{w,k}[$ by quasi-compact open with the property that $\overline{]X_{w,k}[_n} \subseteq]X_{w,k}[_{n+1}$. We deduce that $]X_{w,k}[= \cup]X_{w,k}[_n K_{p,m',b}$ is a covering with the same properties (note that each $]X_{w,k}[_n K_{p,m',b}$ is a finite union of translates of $]X_{w,k}[_n$). Let us put $U_n = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]X_{w,k}[_n K_{p,m',b})$. Then $U = \cup_n U_n$, each U_n is quasi-compact and $\overline{U_n} \subset U_{n+1}$. Since $\overline{T(U) \cap Z}$ is closed, it is compact in the constructible topology, and there is n such that $\overline{T(U) \cap Z} \subseteq U_n$. We may take $U' = U_n$.

We now proceed to find Z_1. By lemma 3.21,

$$(\pi^{tor}_{HT,K_{p,m',b}})^{-1}\big(]C_{w,k}[_{-\infty,0}\big) \subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]Y_{w,k}[).$$

We let $Z_1 = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(\overline{\cup_{s<\frac{1}{2}}]C_{w,k}[_{\overline{0},s} K_{p,m',b}})$. We now find Z_2. We first observe that $\overline{\cup_{s<\frac{1}{2}}]C_{w,k}[_{\overline{0},s}} \subseteq]C^w_k[\subseteq]X^w_k[$, and that $\overline{]X^w_k[} = \mathrm{sp}^{-1}(X^w_k) \subseteq]Y_{w,k}[$. Since $]X^w_k[K_{p,m',b}$ is a finite union of translates of $]X^w_k[$, we can take

$$Z_2 = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(\overline{]X^w_k[}K_{p,m',b}).$$

□

5.4.3. *Change of support condition.* It is important to us that the cohomology $\mathrm{R}\Gamma_w(K^pK_p, \kappa)^{\pm,fs}$ can actually be realized as the finite slope part of cohomology groups with different support conditions. The following definition is motivated by Lemma 2.21 and the discussion of Section 5.3, especially Corollary 5.9. We start by fixing an element $t \in T^{++}$ such that $\min(t) \geq 1$. We let $C = \max(t)$.

Definition 5.11. *Let $m' \geq b \geq 0$ with $m' > 0$. A $(+, w, K_{p,m',b})$-allowed support is a pair $(\mathcal{U}, \mathcal{Z})$ where:*

1. *$\mathcal{U}$ is an open subset of $\mathcal{S}^{tor}_{K^pK_{p,m',b},\Sigma}$ which is a finite union of quasi-Stein open subsets.*
2. *$\mathcal{Z}$ is a closed subset of $\mathcal{S}^{tor}_{K^pK_{p,m',b},\Sigma}$ whose complement is a finite union of quasi-Stein open subsets.*
3. *There exists $m, n, s \in \mathbb{Z}_{\geq 0}$ such that:*

$$(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{m,\overline{0}}K_{p,m',b} \cap]C_{w,k}[_{0,\overline{n+s}}K_{p,m',b}) \subseteq \mathcal{Z} \subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{0,C\overline{n}}K_{p,m',b}),$$

$$(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{m+s,\overline{0}}K_{p,m',b} \cap]C_{w,k}[_{0,\overline{n}}K_{p,m',b}) \subseteq \mathcal{U} \subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{Cm,-1}K_{p,m',b}).$$

Let $m' \geq b \geq 0$ with $m' > 0$. A $(-, w, K_{p,m',b})$-allowed support is a pair $(\mathcal{U}, \mathcal{Z})$ where:

1. *$\mathcal{U}$ is an open subset of $\mathcal{S}^{tor}_{K^pK_{p,m',b},\Sigma}$ which is a finite union of quasi-Stein open subsets.*
2. *$\mathcal{Z}$ is a closed subset of $\mathcal{S}^{tor}_{K^pK_{p,m',b},\Sigma}$ whose complement is a finite union of quasi-Stein open subsets.*

(3) *There exists* $m, n, s \in \mathbb{Z}_{\geq 0}$ *such that:*

$$(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{m+s},0}K_{p,m',b}\cap]C_{w,k}[_{\overline{0},n}K_{p,m',b}) \subseteq \mathcal{Z} \subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{C\overline{m},0}K_{p,m',b}),$$

$$(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{m},0}K_{p,m',b}\cap]C_{w,k}[_{\overline{0},n+s}K_{p,m',b}) \subseteq \mathcal{U} \subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{-1,Cn}K_{p,m',b}).$$

Example 5.12. For any $m' \geq b \geq 0$ with $m' > 0$ and any $s \geq 0$, the pair

$$\mathcal{U} = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,-1}K_{p,m',b}), \quad \mathcal{Z} = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{0},\overline{s}}K_{p,m',b})$$

is a $(+, w, K_{p,m',b})$-allowed support and the pair

$$\mathcal{U} = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{-1,s}K_{p,m',b}), \quad \mathcal{Z} = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{s},\overline{0}}K_{p,m',b})$$

is a $(-, w, K_{p,m',b})$-allowed support.

Theorem 5.13. *Let* $m' \geq b \geq 0$ *with* $m' > 0$*, and fix* $w \in {}^MW$ *and* $\kappa \in X^\star(T^c)^{M_\mu,+}$.

(1) *Let* $(\mathcal{U}, \mathcal{Z})$ *be a* $(+, w, K_{p,m',b})$*-allowed support condition. Then* $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa)$ *and* $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa(-D))$ *are objects of* $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$ *and carry a canonically defined, potent compact action of* T^s*. Moreover there are canonical isomorphisms*

$$\mathrm{R}\Gamma_w(K^pK_{p,m',b}, \kappa)^{+,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa)^{T^s-fs}$$

and

$$\mathrm{R}\Gamma_w(K^pK_{p,m',b}, \kappa, cusp)^{+,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa(-D))^{T^s-fs}.$$

(2) *Let* $(\mathcal{U}, \mathcal{Z})$ *be a* $(-, w, K_{p,m',b})$*-allowed support condition. Then* $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa)$ *and* $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa(-D))$ *are objects of* $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$ *and carry a canonically defined, potent compact action of* T^s*. Moreover there are canonical isomorphisms*

$$\mathrm{R}\Gamma_w(K^pK_{p,m',b}, \kappa)^{-,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa)^{T^s-fs}$$

and

$$\mathrm{R}\Gamma_w(K^pK_{p,m',b}, \kappa, cusp)^{-,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa(-D))^{T^s-fs}.$$

Proof. We will only prove the first point for non cuspidal cohomology. It follows from lemma 2.21 that $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}_\kappa)$ is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(F))$.

For the rest of the statement we will use the infinite diamond construction of section 5.3.2 and corollary 5.9. For $m, n \in \mathbb{Z}_{\geq 0}$ we let $U_m = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(T^m(]X_{w,k}[))$ and $Z_n = (\pi^{tor}_{HT,K_{p,m',b}})^{-1}((T^t)^n(\overline{]Y_{w,k}[}))$. Since we have already checked in the proof of theorem 5.10 that the "restriction-corestriction" map

$$\mathrm{R}\Gamma_{U_0\cap Z_1}(U_0, \mathcal{V}_\kappa) \to \mathrm{R}\Gamma_{U_1\cap Z_0}(U_1, \mathcal{V}_\kappa)$$

is compact, it follows that the conclusion of corollary 5.9 hold.

By lemma 3.29 we have:

$$\begin{aligned}
U_m \cap Z_{n+s} &\subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{m,\overline{0}}K_{p,m',b}\cap]C_{w,k}[_{0,\overline{n+s}}K_{p,m',b}) \\
U_{m+s} \cap Z_n &\subseteq (\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{m+s,\overline{0}}K_{p,m',b}\cap]C_{w,k}[_{0,\overline{n}}K_{p,m',b}) \\
(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{mC,-1}K_{p,m',b}) &\subseteq U_m \\
(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{0,\overline{n}C}K_{p,m',b}) &\subseteq Z_n.
\end{aligned}$$

It follows from corollary 5.9 that if $(\mathcal{U},\mathcal{Z})$ is a $(+,w,K_{p,m',b})$-allowed support condition, then $\mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U},\mathcal{V}_\kappa)$ carries a canonical action of T^s and we have canonical quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U},\mathcal{V}_\kappa)^{T^s-fs}\simeq \mathrm{R}\Gamma_{U\cap Z}(U,\mathcal{V}_\kappa)^{T-fs}\simeq \mathrm{R}\Gamma_w(K^pK_{p,m',b},\kappa)^{+,fs}.$$

□

5.4.4. *Change of level.* Now we investigate how the finite slope cohomologies $\mathrm{R}\Gamma_w(K^pK_p,\kappa)^{\pm,fs}$ and $\mathrm{R}\Gamma_w(K^pK_p,\kappa,cusp)^{\pm,fs}$ vary with the level K_p.

Theorem 5.14. (1) *For all $w\in {}^MW$ and all $m''\geq m'\geq b\geq 0$ with $m'>0$, the pullback map*

$$\mathrm{R}\Gamma_w(K^pK_{p,m',b},\kappa)^{+,fs}\to \mathrm{R}\Gamma_w(K^pK_{p,m'',b},\kappa)^{+,fs}$$

and the trace map

$$\mathrm{R}\Gamma_w(K^pK_{p,m'',b},\kappa)^{-,fs}\to \mathrm{R}\Gamma_w(K^pK_{p,m',b},\kappa)^{-,fs}$$

are quasi-isomorphisms, compatible with the action of $\mathbb{Q}[T(\mathbb{Q}_p)/T_b]$, and the same statements are true for cuspidal cohomology.

(2) *For all $w\in {}^MW$ and all $m'\geq b'\geq b\geq 0$ with $m'>0$, the pullback map*

$$\mathrm{R}\Gamma_w(K^pK_{p,m',b},\kappa)^{+,fs}\to (\mathrm{R}\Gamma_w(K^pK_{p,m',b'},\kappa)^{+,fs})^{T_b/T_{b'}}$$

and the trace map

$$(\mathrm{R}\Gamma_w(K^pK_{p,m',b'},\kappa)^{-,fs})^{T_b/T_{b'}}\to \mathrm{R}\Gamma_w(K^pK_{p,m',b},\kappa)^{-,fs}$$

are quasi-isomorphisms, compatible with the action of $\mathbb{Q}[T(\mathbb{Q}_p)/T_b]$, and the same statements are true for cuspidal cohomology.

Proof. This follows from lemma 4.17. □

As a result of the theorem, we can let $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{\pm,fs}$ and $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{\pm,fs}$ denote $\mathrm{R}\Gamma_w(K^pK_{p,m',b},\kappa)^{\pm,fs}[\chi]$ and $\mathrm{R}\Gamma_w(K^pK_{p,m',b},\kappa,cusp)^{\pm,fs}[\chi]$ for any $m'\geq b\geq \mathrm{cond}(\chi)$ with $m'>0$, as these spaces have been canonically identified.

5.5. **The spectral sequence associated with the Bruhat stratification.** Recall that there is a length function $\ell: {}^MW\to[0,d]$ where $d=\dim\mathcal{FL}=\dim S_K$ with the property that $\ell(w)=\dim C_w$. We let $\ell_+(w)=\ell(w)$ and $\ell_-(w)=d-\ell(w)$.

The main result of this section is the following theorem:

Theorem 5.15. *Let $\kappa\in X^\star(T^c)^{M_\mu,+}$ be a weight and let $\chi: T(\mathbb{Z}_p)\to F^\times$ be a finite order charater. For a choice of $+$ or $-$, there is a $\mathcal{H}^\pm_{p,m,b}$-equivariant spectral sequence $\mathbf{E}^{p,q}(K^p,\kappa,\chi)^\pm$ converging to classical finite slope cohomology $\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs}$, such that*

$$\mathbf{E}_1^{p,q}(K^p,\kappa,\chi)^\pm=\oplus_{w\in {}^MW,\ell_\pm(w)=p}\mathrm{H}^{p+q}_w(K^p,\kappa,\chi)^{\pm,fs}.$$

There are also spectral sequences $\mathbf{E}^{p,q}(K^p,\kappa,\chi,cusp)^\pm$ converging to $\mathrm{H}^{p+q}(K^p,\kappa,\chi,cusp)^{\pm,fs}$ such that

$$\mathbf{E}_1^{p,q}(K^p,\kappa,\chi,cusp)^\pm=\oplus_{w\in {}^MW,\ell_\pm(w)=p}\mathrm{H}^{p+q}_w(K^p,\kappa,\chi,cusp)^{\pm,fs}.$$

5.5.1. *Construction of a filtration.* We consider the following two stratifications of the special fiber of the flag variety FL_k, the first one is by open subsets:

$$\{FL_k^{\geq r} = \cup_{\ell(w)\geq r} C_{w,k}\}_{0\leq r\leq d}$$

and the second one is by closed subsets:

$$\{FL_k^{\leq d-r} = \cup_{\ell(w)\leq d-r} C_{w,k}\}_{0\leq r\leq d}.$$

We then define $\overline{]FL_k^{\geq r}[} := Z_r^+$ and $\overline{]FL_k^{\leq d-r}[} := Z_r^-$. This gives two filtrations

$$\mathcal{FL} = Z_0^{\pm} \supset Z_1^{\pm} \supset \cdots \supset Z_d^{\pm} \supset Z_{d+1}^{\pm} = \emptyset$$

by closed subspaces invariant under Iw.

Now let $K_p = K_{p,m,b}$ for some $m \geq b \geq 0$ with $m > 0$. We can consider the pullback of these filtrations by π^{tor}_{HT,K_p} to get two filtrations $\mathcal{S}^{tor}_{K,\Sigma} = \mathcal{Z}_0^{\pm} \supset \mathcal{Z}_1^{\pm} \supset \cdots \supset \mathcal{Z}_d^{\pm} \supset \mathcal{Z}_{d+1}^{\pm} = \emptyset$ by closed subspaces.

For any weight κ, we can consider the associated spectral sequence (see section 2.3):

$$\mathrm{H}^{p+q}_{\mathcal{Z}_p^{\pm}/\mathcal{Z}_{p+1}^{\pm}}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa) \Rightarrow \mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa).$$

By definition,

$$\mathrm{H}^{\star}_{\mathcal{Z}_p^{+}/\mathcal{Z}_{p+1}^{+}}(\mathcal{S}^{tor}_{K^pK_p}, \mathcal{V}_\kappa) =$$
$$\mathrm{H}^{\star}_{(\pi^{tor}_{HT,K_p})^{-1}(\overline{]\cup_{\ell(w)\geq p} C_w[}\cap]\cup_{\ell(w)\leq p} C_w[)}((\pi^{tor}_{HT,K_p})^{-1}(]\cup_{\ell(w)\leq p} C_w[), \mathcal{V}_\kappa).$$

and

$$\mathrm{H}^{\star}_{\mathcal{Z}_p^{-}/\mathcal{Z}_{p+1}^{-}}(\mathcal{S}^{tor}_{K^pK_p}, \mathcal{V}_\kappa) =$$
$$\mathrm{H}^{\star}_{(\pi^{tor}_{HT,K_p})^{-1}(\overline{]\cup_{\ell(w)\leq d-p} C_w[}\cap]\cup_{\ell(w)\geq d-p} C_w[)}((\pi^{tor}_{HT,K_p})^{-1}(]\cup_{\ell(w)\geq d-p} C_w[), \mathcal{V}_\kappa).$$

We also have a cuspidal version

$$\mathrm{H}^{p+q}_{\mathcal{Z}_p^{\pm}/\mathcal{Z}_{p+1}^{\pm}}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa(-D)) \Rightarrow \mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa(-D)).$$

We relate the $\mathbf{E}_1$ pages of these spectral sequences to the overconvergent cohomologies considered in the previous section.

Lemma 5.16. *For all p,*

$$\mathrm{R}\Gamma_{\mathcal{Z}_p^{+}/\mathcal{Z}_{p+1}^{+}}(\mathcal{S}^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_\kappa) =$$
$$\oplus_{w\in {}^MW,\ \ell(w)=p} \mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[\cap\overline{]Y_{w,k}[})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}_\kappa)$$
$$\mathrm{R}\Gamma_{\mathcal{Z}_p^{-}/\mathcal{Z}_{p+1}^{-}}(\mathcal{S}^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_\kappa) =$$
$$\oplus_{w\in {}^MW,\ \ell(w)=d-p} \mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[\cap\overline{]X_{w,k}[})}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[), \mathcal{V}_\kappa)$$

We have similar results for cuspidal cohomology.

Proof. We have

$$\overline{]\cup_{\ell(w)\geq p} C_w[} \cap]\cup_{\ell(w)\leq p} C_w[= \cup_{w,\ell(w)=p} \overline{]Y_{w,k}[} \cap]X_{w,k}[$$

and

$$\overline{]\cup_{\ell(w)\leq d-p} C_w[} \cap]\cup_{\ell(w)\geq d-p} C_w[= \cup_{w,\ell(w)=d-p}]Y_{w,k}[\cap \overline{]X_{w,k}[}$$

by lemma 3.20. The conclusion follows from lemma 2.1. ∎

Lemma 5.17. *For a choice of* $+$ *or* $-$, $\mathcal{H}^{\pm}_{p,m,b}$ *acts on* $\mathrm{R}\Gamma_{\mathcal{Z}_p^{\pm}/\mathcal{Z}_{p+1}^{\pm}}(\mathcal{S}^{tor}_{K^pK_p,\Sigma},\mathcal{V}_\kappa)$, *and the spectral sequence*

$$\mathrm{H}^{p+q}_{\mathcal{Z}_p^{\pm}/\mathcal{Z}_{p+1}^{\pm}}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{V}_\kappa)\Rightarrow \mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{V}_\kappa)$$

is $\mathcal{H}^{\pm}_{p,m,b}$*-equivariant. The same result holds for cuspidal cohomology.*

Proof. Easy and left to the reader. □

5.5.2. *Proof of Theorem 5.15.* For a choice of $\pm$, we have constructed a spectral sequence

$$\mathrm{H}^{p+q}_{\mathcal{Z}_p^{\pm}/\mathcal{Z}_{p+1}^{\pm}}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{V}_\kappa)\Rightarrow \mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{V}_\kappa).$$

The Hecke algebra $\mathcal{H}^{\pm}_{p,m,b}$ acts on this spectral sequence and it makes sense to take the finite slope part by lemma 5.17. Applying the finite slope projector, we obtain the spectral sequence of the theorem.

5.5.3. *Cousin complexes.* We now extract from the spectral sequence certain complexes that play a prominent role, in particular in view of conjecture 5.20. We let $\mathcal{C}ous(K^p,\kappa,\chi)^{\pm}$ be the complex $\mathbf{E}_1^{\bullet,0}(K^p,\kappa,\chi)^{\pm}$ (w_0^M is the longest element of ${}^MW$):

$$\mathrm{H}^0_{Id/w_0^M}(K^p,\kappa,\chi)^{\pm,fs}\to\oplus_{w\in{}^MW,\ell_{\pm}(w)=1}\mathrm{H}^1_w(K^p,\kappa,\chi)^{\pm,fs}\to$$
$$\oplus_{w\in{}^MW,\ell_{\pm}(w)=2}\mathrm{H}^2_w(K^p,\kappa,\chi)^{\pm,fs}\to\cdots\to\mathrm{H}^d_{w_0^M/Id}(K^p,\kappa,\chi)^{\pm,fs}$$

and we let $\mathcal{C}ous(K^p,\kappa,\chi,cusp)^{\pm}$ be the complex $\mathbf{E}_1^{\bullet,0}(K^p,\kappa,\chi,cusp)^{\pm}$:

$$\mathrm{H}^0_{Id/w_0^M}(K^p,\kappa,\chi,cusp)^{\pm,fs}\to\oplus_{w\in{}^MW,\ell_{\pm}(w)=1}\mathrm{H}^1_w(K^p,\kappa,\chi,cusp)^{\pm,fs}\to$$
$$\oplus_{w\in{}^MW,\ell_{\pm}(w)=2}\mathrm{H}^2_w(K^p,\kappa,\chi,cusp)^{\pm,fs}\to\cdots\to\mathrm{H}^d_{w_0^M/Id}(K^p,\kappa,\chi,cusp)^{\pm,fs}$$

5.6. **Cohomological vanishing.** The following vanishing theorem is crucial in order to study the spectral sequence.

Theorem 5.18. *The cohomology complex* $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{\pm,fs}$ *has amplitude* $[0,\ell_{\pm}(w)]$.

Proof. We only give the argument for the $+$ case, as the $-$ case follows with minor modifications. Let b be the conductor of χ. We can realize $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$ as the χ-isotypic part of the finite slope part of

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,-1}K_{p,m',b}\cap C_{w,k}[_{\overline{0},\overline{s}}K_{p,m',b})}((\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,-1}K_{p,m',b}),\mathcal{V}_\kappa(-D))$$

for any $s\geq 0$ and $m'\geq b$, $m'>0$ by example 5.12 and theorem 5.13.

We fix s large enough so that $\pi^{-1}_{HT}(]C_{w,k}[_{s,s-1})$ is quasi-Stein in the minimal compactification. We also fix $m'=s$. To simplify notations, we let $K_p=K_{p,m',b}$.

We observe that under these assumptions, $]C_{w,k}[_{s,-1}K_p\cap]C_{w,k}[_{\overline{0},\overline{s}}K_p=]C_{w,k}[_{s,\overline{s}}K_p$ by lemma 3.19, and therefore the above cohomology writes

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{s,\overline{s}}K_p)}((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{s,s-1}K_p),\mathcal{V}_\kappa(-D)).$$

We shall prove that

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{s,\overline{s}}K_p)}((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{s,s-1}K_p),\mathcal{V}_\kappa(-D))$$

has cohomological amplitude $[0,\ell_+(w)]$.

Le K'_p be the principal level m' congruence subgroup. Since

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{s,\overline{s}}K_p)}((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{s,s-1}K_p),\mathcal{V}_\kappa(-D)) =$$

$$\mathrm{R}\Gamma\big(K_p/K'_p, \mathrm{R}\Gamma_{(\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,\overline{s}}K_p)}((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}K_p),\mathcal{V}_\kappa(-D))\big)$$

It will suffice to prove that

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,\overline{s}}K_p)}((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}K_p),\mathcal{V}_\kappa(-D))$$

has cohomological amplitude in $[0,\ell_+(w)]$.

Since $]C_{w,k}[_{s,\overline{s}}K_p$ is a finite disjoint union of translates of $\{]C_{w,k}[_{s,\overline{s}}k_i\}$ for elements $k_1,\cdots,k_n$ by lemma 3.18, we are left to prove that

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,\overline{s}}k_i)}((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}k_i),\mathcal{V}_\kappa(-D))$$

for $1\leq i\leq n$ has cohomological amplitude in $[0,\ell_+(w)]$. Also, using the action of K_p we may assume that $k_i=1$. The cohomology fits in a triangle:

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,\overline{s}})}((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}),\mathcal{V}_\kappa(-D))\to$$

$$\mathrm{R}\Gamma((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}),\mathcal{V}_\kappa(-D))\to$$

$$\mathrm{R}\Gamma((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}\setminus]C_{w,k}[_{s,\overline{s}}),\mathcal{V}_\kappa(-D))\stackrel{+1}{\to}$$

Since $\pi^{-1}_{HT,K'_p}(]C_{w,k}[_{s,s-1})$ is quasi-Stein in the minimal compactification $\mathcal{S}^\star_{K^pK'_p}$ we have (by theorem 4.6)

$$\mathrm{R}\Gamma((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}),\mathcal{V}_\kappa(-D))=\mathrm{R}\Gamma(\pi^{-1}_{HT,K'_p}(]C_{w,k}[_{s,s-1}),(\pi_{K,\Sigma})_\star\mathcal{V}_\kappa(-D))$$

is concentrated in degree 0.

We will now prove that $\pi^{-1}_{HT,K'_p}(]C_{w,k}[_{s,s-1}\backslash]C_{w,k}[_{s,\overline{s}})$ admits a covering by $\ell_+(w)$ acyclic spaces. This will show that

$$\mathrm{R}\Gamma((\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{s,s-1}\setminus]C_{w,k}[_{s,\overline{s}}),\mathcal{V}_\kappa(-D))=$$

$$\mathrm{R}\Gamma(\pi^{-1}_{HT,K'_p}(]C_{w,k}[_{s,s-1}\setminus]C_{w,k}[_{s,\overline{s}}),(\pi_{K^pK'_p,\Sigma})_\star\mathcal{V}_\kappa(-D))$$

has only cohomology in degree 0 to $\ell_+(w)-1$ and the theorem will follow.

We recall from corollary 3.12 the isomorphism:

$$\begin{array}{rcl}\prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^+}\mathcal{U}_\alpha\times\prod_{\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^-}\mathcal{U}^o_\alpha & \to &]C_{w,k}[_{\mathcal{FL}}\\ (u_\alpha)_{\alpha\in w^{-1}\Phi^{-,M}} & \mapsto & w\prod_\alpha u_\alpha\end{array}$$

Let us fix coordinates $1+u_\alpha$ on each of the one parameter groups. For these coordinates, the equation of $]C_{w,k}[_{s,s-1}\backslash]C_{w,k}[_{s,\overline{s}}$ is:

- $\forall\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^+$, $|u_\alpha|\leq|p^{s-1}|$,
- $\forall\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^-$, $\exists\epsilon>0$, $|u_\alpha|\leq|p^{s+\epsilon}|$,
- $\exists\alpha\in(w^{-1}\Phi^{-,M})\cap\Phi^+$, $\exists\nu>0$, $|u_\alpha|\geq|p^{s-\nu}|$.

Since $\sharp(w^{-1}\Phi^{-,M})\cap\Phi^+=\ell(w)=\ell_+(w)$, we deduce that $]C_{w,k}[_{s,s-1}\setminus]C_{w,k}[_{s,\overline{s}}$ is indeed covered by $\ell(w)$ acyclic spaces and the same holds for its pre-image by π_{HT,K'_p}. □

Proposition 5.19. *For the spectral sequence* $\mathbf{E}^{p,q}(K^p, \kappa, \chi, cusp)^{\pm}$ *converging to* $\mathrm{H}^{p+q}(K^p, \kappa, \chi, cusp)^{\pm, fs}$ *we have* $\mathbf{E}_1^{p,q}(K^p, \kappa, \chi, cusp)^{\pm} = 0$ *if* $q > 0$. *In particular* $\mathbf{E}_\infty^{p,q}(K^p, \kappa, \chi, cusp)^{\pm} = \mathrm{Gr}^p(\mathrm{H}^{p+q}(K^p, \kappa, \chi, cusp)^{\pm, fs}) = 0$ *if* $p < p + q$.

Proof. This follows from theorem 5.18. The spectral sequence pictures as follows (w_0^M is the longest element of ${}^M W$), where the top horizontal line is the Cousin complex:

$$\begin{array}{llll}
\mathrm{H}^0_{Id/w_0^M}(K^p,\kappa,\chi,cusp)^{\pm,fs} & \oplus_{w\in {}^MW,\ell_\pm(w)=1}\mathrm{H}^1_w(K^p,\kappa,\chi,cusp)^{\pm,fs} & \oplus_{w\in {}^MW,\ell_\pm(w)=2}\mathrm{H}^2_w(K^p,\kappa,\chi,cusp)^{\pm,fs} & \cdots \\
 & \oplus_{w\in {}^MW,\ell_\pm(w)=1}\mathrm{H}^0_w(K^p,\kappa,\chi,cusp)^{\pm,fs} & \oplus_{w\in {}^MW,\ell_\pm(w)=2}\mathrm{H}^1_w(K^p,\kappa,\chi,cusp)^{\pm,fs} & \cdots \\
 & & \oplus_{w\in {}^MW,\ell_\pm(w)=2}\mathrm{H}^0_w(K^p,\kappa,\chi,cusp)^{\pm,fs} & \cdots
\end{array}$$

□

We now conjecture the opposite vanishing threorem for the non-cuspidal cohomology:

Conjecture 5.20. *The cohomology complex* $\mathrm{R}\Gamma_w(K^p, \kappa, \chi)^{\pm, fs}$ *has amplitude* $[\ell_\pm(w), d]$. *In particular, if the Shimura variety is compact, the spectral sequence is concentrated in the indices* (p, q) *with* $q = 0$ *and the cousin complex* $\mathcal{C}ous(K^p, \kappa, \chi)^{\pm}$ *(see section 5.5.3) computes* $\mathrm{R}\Gamma(K^p, \kappa, \chi)^{\pm, fs}$.

See proposition 5.25 and corollary 5.26 for a partial confirmation of this conjecture. Here is a proof of the conjecture for compact Shimura varieties of dimension ≤ 2

Proposition 5.21. *Conjecture 5.20 holds when* $\ell_\pm(w) \leq 1$, *and also when* $\ell_{\pm(w)} = d$ *if the Shimura variety is compact. In particular 5.20 holds for compact Shimura varieties of dimension* ≤ 2.

Proof. The case $\ell_\pm(w) = 0$ is vacuous and the case $\ell_\pm(w) = 1$ follows from the more general observation that when $\ell_\pm(w) > 0$, $\mathrm{H}^0_w(K^p, \kappa, \chi)^{\pm, fs} = 0$, as this is computed as the space of sections of a vector bundle supported on a proper closed subset of each irreducible component.

When the Shimura variety is compact and $\ell_\pm(w) = d$, $\mathrm{R}\Gamma_w(K^p, \kappa, \chi)^{\pm, fs}$ is computed as the finite slope part of the cohomology with compact support of a vector bundle on a smooth affinoid, and hence vanishes below degree d by theorem 2.32. □

5.7. **Duality.** In this section we investigate Serre duality on overconvergent cohomologies. It turns out that combining a weak form of duality with theorem 5.18 we can obtain some partial results towards conjecture 5.20.

Theorem 5.22. *For all* $w \in {}^M W$, *there is a pairing:*

$$\langle , \rangle : \mathrm{H}^i_w(K_p, \kappa, \chi, cusp)^{\pm, fs} \times \mathrm{H}^{d-i}_w(K_p, -2\rho_{nc} - w_{0,M}\kappa, \chi^{-1})^{\mp, fs} \to F$$

This pairing induces a pairing between the spectral sequences:

$$\langle , \rangle_{p,q,r} : \mathbf{E}_r^{p,q}(K^p, \kappa, \chi, cusp)^{\pm} \times \mathbf{E}_r^{d-p,-q}(K^p, -2\rho_{nc} - w_{0,M}\kappa, \chi^{-1})^{\mp} \to F$$

On the abutment of the spectral sequence the pairing $\langle,\rangle_{p,q,\infty}$ *is induced by the perfect Serre duality pairing:*

$$\mathrm{H}^{p+q}(K^p,\kappa,\chi,cusp)^{\pm,fs}\times \mathrm{H}^{d-p-q}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1})^{\mp,fs}\to F.$$

Proof. We construct the pairing

$$\langle,\rangle:\mathrm{H}^i_w(K_p,\kappa,\chi,cusp)^{+,fs}\times \mathrm{H}^{d-i}_w(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1})^{-,fs}\to F.$$

Let b be the conductor of χ. We can realize $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{+,fs}$ as the χ-isotypic part of the finite slope part of

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,\overline{s}}K_{p,m',b})}((\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,-1}K_{p,m',b}),\mathcal{V}_\kappa(-D))$$

for any $s>0$ and $m'\geq b$, $m'>s$. by example 5.12 and theorem 5.13.

We can realize $\mathrm{R}\Gamma_w(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1})^{-,fs}$ as the χ^{-1}-isotypic part of the finite slope part of

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{s+1},s-1}K_{p,m',b})}((\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{-1,s-1}K_{p,m',b}),\mathcal{V}_{-2\rho_{nc}-w_{0,M}\kappa}).$$

We have a cup-product by proposition 2.3:

$$\mathrm{H}^i_{(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,\overline{s}}K_{p,m',b})}((\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,-1}K_{p,m',b}),\mathcal{V}_\kappa(-D))\times$$

$$\mathrm{H}^{d-i}_{(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{s+1},s-1}K_{p,m',b})}((\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{-1,s-1}K_{p,m',b}),\mathcal{V}_{-2\rho_{nc}-w_{0,M}\kappa})$$

$$\to \mathrm{H}^d_{(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{s+1},\overline{s}}K_{p,m',b})}((\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,s-1}K_{p,m',b},\mathcal{V}_{-2\rho_{nc}}(-D))$$

and there is a trace map (by theorem 2.32):

$$\mathrm{H}^d_{(\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{\overline{s+1},\overline{s}}K_{p,m',b})}((\pi^{tor}_{HT,K_{p,m',b}})^{-1}(]C_{w,k}[_{s,s-1}K_{p,m',b}),\mathcal{V}_{-2\rho_{nc}}(-D))\to F.$$

This pairing intertwines the actions of $\mathcal{H}^+_{p,m',b}$ and $\mathcal{H}^-_{p,m',b}$. It is straightforward (but painful) to check that the induced pairing

$$\langle,\rangle:\mathrm{H}^i_w(K_p,\kappa,\chi,cusp)^{+,fs}\times \mathrm{H}^{d-i}_w(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1})^{-,fs}\to F$$

is independent of choices.

The rest of the theorem follows from the functoriality of the trace map.

□

We make the following conjecture:

Conjecture 5.23. *The pairing*

$$\langle,\rangle:\mathrm{H}^i_w(K_p,\kappa,\chi,cusp)^{\pm,fs}\times \mathrm{H}^{d-i}_w(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1})^{\mp,fs}\to F$$

is non degenerate. (Equivalently it induces perfect pairings between the finite dimensional generalized eigenspaces for $\mathbb{Z}[T^\pm]$*).*

Remark 5.24. Conjecture 5.23 implies conjecture 5.20. We also note that this conjecture holds for compact Shimura varieties when $\ell_\pm(w)=0$ by theorem 2.32, (5).

The following proposition is a partial confirmation of conjecture 5.20:

Proposition 5.25. *For the spectral sequence* $\mathbf{E}^{p,q}(K^p,\kappa,\chi)^\pm$ *converging to* $\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs}$ *we have* $\mathbf{E}^{p,q}_\infty(K^p,\kappa,\chi)^\pm=\mathrm{Gr}^p(\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs})=0$ *if* $p>p+q$.

Proof. We have a perfect pairing $\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs}\times\mathrm{H}^{d-p-q}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,fs}\to F$. Because we have a pairing at the level of the spectral sequences degenerating to this pairing, we deduce that

$$\mathrm{Fil}^{p+1}\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs}\subseteq\big(\mathrm{Fil}^{d-p}\mathrm{H}^{d-p-q}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,fs}\big)^{\perp}$$

We know by proposition 5.19 that $\mathrm{Fil}^{d-p}\mathrm{H}^{d-p-q}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,fs}=\mathrm{H}^{d-p-q}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,fs}$ for $d-p\leq d-p-q$ and therefore, $\mathrm{Fil}^{p+1}\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs}=0$ for $p\geq p+q$. □

The following corollary illustrates the importance of the Cousin complex:

Corollary 5.26. *If the Shimura variety is compact, for the spectral sequence* $\mathbf{E}^{p,q}(K^p,\kappa,\chi)^{\pm}$ *converging to* $\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs}$ *we have* $\mathbf{E}_{\infty}^{p,q}(K^p,\kappa,\chi)^{\pm}=0$ *if* $q\neq 0$*. In particular, for all* i*,* $\mathrm{H}^i(K^p,\kappa,\chi)^{\pm,fs}$ *is a subquotient of* $\mathrm{H}^i(\mathcal{C}ous(K^p,\kappa,\chi)^{\pm})$*.*

5.8. **Interior cohomology.** For non compact Shimura varieties, we can introduce the interior cohomology:

$$\overline{\mathrm{H}}^i(K^p,\kappa,\chi)^{\pm,fs}:=\mathrm{Im}(\mathrm{H}^i(K^p,\kappa,\chi,cusp)^{\pm,fs}\to\mathrm{H}^i(K^p,\kappa,\chi)^{\pm,fs}).$$

We can also consider some “interior” version of the Cousin complex. We first let $\mathcal{C}ous(K^p,\kappa,\chi)^{\pm,\vee}$ be the complex:

$$\mathrm{H}^d_{w_0^M/Id}(K^p,\kappa,\chi)^{\pm,fs,\vee}\to\oplus_{w\in{}^MW,\ell_{\pm}(w)=d-1}\mathrm{H}^{d-1}_w(K^p,\kappa,\chi)^{\pm,fs,\vee}\to$$

$$\oplus_{w\in{}^MW,\ell_{\pm}(w)=d-2}\mathrm{H}^{d-2}_w(K^p,\kappa,\chi)^{\pm,fs,\vee}\to\cdots\to\mathrm{H}^0_{Id/W_0^M}(K^p,\kappa,\chi)^{\pm,fs,\vee}$$

where $\mathrm{H}^i_w(K^p,\kappa,\chi)^{\pm,fs,\vee}$ is the vector space of linear forms $L\in\mathrm{Hom}_F(\mathrm{H}^i_w(K^p,\kappa,\chi)^{\pm,fs,\vee},F)$ such that for there exists $h\in\mathbb{Q}$ with $L(\mathrm{H}^i_w(K^p,\kappa,\chi)^{\pm,>h})=0$. We define similarly $\mathcal{C}ous(K_p,\kappa,\chi,cusp)^{\pm,fs,\vee}$.

By the pairing of theorem 5.22, we have maps

$$\mathcal{C}ous(K^p,\kappa,\chi,cusp)^{\pm}\to\mathcal{C}ous(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1})^{\mp,\vee}$$

and

$$\mathcal{C}ous(K^p,\kappa,\chi)^{\pm}\to\mathcal{C}ous(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,\vee}.$$

We conjecture (conjecture 5.23) that these maps are isomorphisms of complexes.

We define the interior Cousin complex

$$\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^{\pm}=$$

$$\mathrm{Im}\big(\mathcal{C}ous(K^p,\kappa,\chi,cusp)^{\pm}\to\mathcal{C}ous(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,\vee}\big).$$

For all $w\in{}^MW$, let $\overline{\mathrm{H}}^{\star}_w(K^p,\kappa,\chi)^{\pm,fs}=\mathrm{Im}\big(\mathrm{H}^i_w(K^p,\kappa,\chi,cusp)^{\pm}\to\mathrm{H}^i_w(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,\vee}\big)$.

By definition, $\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^{\pm}$ is concentrated in degrees in the interval $[0,d]$ and its degree i object is $\oplus_{w\in{}^MW,\ell_{\pm}(w)=i}=\overline{\mathrm{H}}^i_w(K^p,\kappa,\chi)^{\pm,fs}$.

Corollary 5.27. *We have the formula:*

$$\overline{\mathrm{H}}^p(K^p,\kappa,\chi)^{\pm,fs}=\mathrm{Im}(\mathbf{E}_{\infty}^{p,0}(K^p,\kappa,\chi,cusp)^{\pm}\to\mathbf{E}_{\infty}^{d-p,0}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,\vee}).$$

In particular, for all i*,* $\overline{\mathrm{H}}^i(K^p,\kappa,\chi)^{\pm,fs}$ *is a subquotient of* $\mathrm{H}^i(\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^{\pm})$*.*

Proof. The spectral sequence $\mathbf{E}^{p,q}(K^p,\kappa,\chi,cusp)^{\pm}$ converges to $\mathrm{H}^{p+q}(K^p,\kappa,\chi,cusp)^{\pm,fs}$. The spectral sequence $\mathbf{E}^{p,q}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp}$ converges to $\mathrm{H}^{p+q}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,fs}=\mathrm{H}^{p+q}(K^p,\kappa,\chi)^{\pm,fs,\vee}$. We have for all $r\in\mathbb{Z}$, maps

$$\mathbf{E}_{\infty}^{p+r,-r}(K^p,\kappa,\chi,cusp)^{\pm}\to\mathbf{E}_{\infty}^{d-p-r,r}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,\vee}$$

and it follows from proposition 5.19, that when $r>0$, $\mathbf{E}_{\infty}^{d-p-r,r}(K^p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,\vee}=0$. We deduce that there is a map: $\mathbf{E}_{\infty}^{p,0}(K^p,\kappa,\chi,cusp)^{\pm}\to\mathrm{H}^p(K^p,\kappa,\chi)^{\pm,fs}$. Since when $r<0$, $\mathbf{E}_{\infty}^{p+r,-r}(K^p,\kappa,\chi,cusp)^{\pm}=0$ it follows that the image of $\mathbf{E}_{\infty}^{p,0}(K^p,\kappa,\chi,cusp)^{\pm}\to\mathrm{H}^p(K^p,\kappa,\chi)^{\pm,fs}$ is $\overline{\mathrm{H}}^p(K^p,\kappa,\chi)^{\pm,fs}$. □

5.9. **Lower bounds on slopes.** In this section we will write $\langle -,-\rangle$ for the usual pairing $X_{\star}(T)\times X^{\star}(T)\to\mathbb{Z}$. We denote by T^d the maximal $\mathbb{Q}_p$-split subtorus of T. We have a relative root system $\Phi_d\subset X^*(T^d)$ with a choice of positive and simple roots $\Delta_d\subset\Phi_d^+\subset\Phi_d$. Because $G/\mathbb{Q}_p$ is quasi-split, restriction from T to T^d defines a surjective map $r:\Phi\to\Phi_d$, which restricts to a surjective map $r:\Delta\to\Delta_d$ (the fibers of r are exactly the Galois orbits of absolute roots.)

We have on $X^{\star}(T)_{\mathbb{R}}$ a partial order $\preceq$ where $\lambda\preceq\lambda'$ if and only if $\lambda'-\lambda\in\mathbb{R}_{\geq0}\Delta$. We have on $X^{\star}(T^d)_{\mathbb{R}}$ a partial order $\leq$ where $\lambda\leq\lambda'$ if and only if $\lambda'-\lambda\in\mathbb{R}_{\geq0}\Delta_d$. We extend the symbol $\leq$ to the case that one or both sides are in $X^{\star}(T)_{\mathbb{R}}$, in which case we apply the restriction map $X^{\star}(T)\to X^{\star}(T^d)$ (so in particular for $\lambda,\lambda'\in X^{\star}(T)$, $\lambda\preceq\lambda'$ implies $\lambda\leq\lambda'$, but not necessarily conversely.)

Recall that we have monoids T^+ and T^- in $T(\mathbb{Q}_p)$. In section 3.5.1, we defined a valuation morphism $v:T(\mathbb{Q}_p)\to X_{\star}(T^d)\otimes\mathbb{Q}$, whose image is a lattice, and whose kernel is the maximal compact subgroup of $T(\mathbb{Q}_p)$ which we denoted by $T(\mathbb{Z}_p)$. For $\lambda\in X^{\star}(T)_{\mathbb{R}}$ and $t\in T(\mathbb{Q}_p)$ we will abusively write $v(\lambda(t))$ for $\langle v(t),\lambda\rangle$. The partial order $\leq$ has another characterization that we frequently use:

Lemma 5.28. *Let $\lambda,\lambda'\in X^{\star}(T^d)_{\mathbb{R}}$. Then $\lambda\leq\lambda'$ if and only if $v(\lambda(t))\leq v(\lambda'(t))$ for all $t\in T^+$.*

Given a homomorphism $\lambda:T(\mathbb{Q}_p)\to\overline{F}^{\times}$, the composition with the valuation $v:\overline{F}^{\times}\to\mathbb{R}$ factors through a morphism $v(T(\mathbb{Q}_p))=T(\mathbb{Q}_p)/T(\mathbb{Z}_p)\to\mathbb{R}$. This extends by linearity to a $\mathbb{R}$-linear map $X_{\star}(T)_{\mathbb{R}}\to\mathbb{R}$ and thus defines an element of $X^{\star}(T)_{\mathbb{R}}$, which we will denote by $v(\lambda)$ and call the *slope* of λ. Unravelling the definition we have $\langle v(\lambda),v(t)\rangle=v(\lambda(t))$.

If we start instead with a monoid homomorphism $T^{\pm}\to\overline{F}^{\times}$, we also define the slope $v(\lambda)$ of λ by first extending λ to a group homomorphism $T(\mathbb{Q}_p)\to\overline{F}^{\times}$ (recall that $T(\mathbb{Q}_p)$ is generated by the monoids $T^{\pm}$).

We now formulate a general conjectural lower bound on the slopes of $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{\pm,fs}$ and $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{\pm,fs}$

Conjecture 5.29. *Fix $w\in{}^MW$, $\kappa\in X^{\star}(T^c)^{M_{\mu},+}$, and $\chi:T(\mathbb{Z}_p)\to F^{\times}$ of finite order.*

(1) *For any character λ of T^+ on $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{+,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{+,fs}$ we have $v(\lambda)\geq w^{-1}w_{0,M}(\kappa+\rho)+\rho$.*
(2) *For any character λ of T^- on $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{-,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{-,fs}$ we have $v(\lambda)\leq w^{-1}(\kappa+\rho)-\rho$.*

Remark 5.30. We can spell out the meaning of these inequalities. The inequality $v(\lambda)\geq w^{-1}w_{0,M}(\kappa+\rho)+\rho$ means that for all $t\in T^+$ (and corresponding $v(t)\in$

$X_\star(T^d)^+_{\mathbb{Q}}$), we have

$$v(\lambda(t)) \geq \langle v(t), w^{-1}w_{0,M}(\kappa+\rho)+\rho\rangle.$$

The inequality $v(\lambda) \leq w^{-1}(\kappa+\rho)-\rho$ means that for all $t \in T^-$ and (and corresponding $v(t) \in X_\star(T^d)^-_{\mathbb{Q}}$), we have

$$v(\lambda(t)) \geq \langle v(t), w^{-1}(\kappa+\rho)-\rho\rangle.$$

Remark 5.31. We recall that for any $w \in W$ we have $\rho+w\rho, \rho-w\rho \in X^\star(T)^+$ (even if ρ is not itself in $X^\star(T)$.) It follows that for all $t \in T^+$ we have $\langle t, \rho+w^{-1}w_{0,M}\rho\rangle \in \mathbb{Z}_{\geq 0}$ and for all $t \in T^-$ we have $-\langle t, \rho - w^{-1}\rho\rangle \in \mathbb{Z}_{\geq 0}$.

Remark 5.32. The bounds of Conjecture 5.29 are compatible with duality in the sense that they are exchanged upon replacing t by t^{-1} and κ by $-2\rho_{nc} - w_{0,M}\kappa$.

On the right hand side of the inequality of conjecture 5.29 we have $w^{-1}w_{0,M}(\kappa+\rho)+\rho$ and $w^{-1}(\kappa+\rho)-\rho$. Each of these expressions can be separated into $(w^{-1}w_{0,M}\kappa) + (w^{-1}w_{0,M}\rho+\rho)$ and $(w^{-1}\kappa)+(w^{-1}\rho-\rho)$ where the first term depends on κ and is related to the action of the Hecke correspondences on the sheaf, while the second term is independent of κ and is related to the geometry of the correspondence (and in particular to the ramification of integral models of the correspondence). The second term is the more delicate to study.

The main result of this section is a bound which is slightly weaker than the conjecture (and concerns only the first term).

Theorem 5.33. *Fix $w \in {}^MW$, $\kappa \in X^\star(T^c)^{M_\mu,+}$, and $\chi : T(\mathbb{Z}_p) \to F^\times$ of finite order.*

1. *For any character λ of T^+ on $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{+,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{+,fs}$ we have $v(\lambda) \geq w^{-1}w_{0,M}\kappa$.*
2. *For any character λ of T^- on $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{-,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,cusp)^{-,fs}$ we have $v(\lambda) \leq w^{-1}\kappa$.*

5.9.1. *Proof of theorem 5.33.* Let $\kappa \in X^\star(T^c)^{M_\mu,+}$. The definition of the sheaf $\mathcal{V}_\kappa$ is given in 4.1.1 with the help of the torsor $\mathcal{M}^{an}_{dR}$ and modeled on the highest weight representation V_κ. By corollary 4.72, the sheaf $\mathcal{V}_\kappa$ has an integral structure $\mathcal{V}^+_\kappa$ (in the sense of definition 2.26), constructed with the help of the $\mathcal{M}_\mu$-torsor $\mathcal{M}_{dR}$ (see proposition 4.69) and modeled on the submodule $V^+_\kappa \subseteq V_\kappa$.

Lemma 5.34. *Let $K_p = K_{p,m',b}$ for $m' \geq b \geq 0$ and $m' > 0$.*

1. *Let $t \in T^+$. For all $n \geq 1$, the isomorphism $p_2^\star\mathcal{V}_\kappa \to p_1^\star\mathcal{V}_\kappa$ induces a map $p_2^\star\mathcal{V}^+_\kappa \to p^{\langle wv(t), w_{0,M}\kappa\rangle}p_1^\star\mathcal{V}^+_\kappa$ on*
$$p_2^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(w\mathcal{G}_nK_p)\big) \cap p_1^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(w\mathcal{G}_nK_p)\big).$$
2. *Let $t \in T^-$. For all $n \geq 1$, the isomorphism $p_2^\star\mathcal{V}_\kappa \to p_1^\star\mathcal{V}_\kappa$ induces a map $p_2^\star\mathcal{V}^+_\kappa \to p^{\langle wv(t), \kappa\rangle}p_1^\star\mathcal{V}^+_\kappa$ on*
$$p_2^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(w\mathcal{G}_nK_p)\big) \cap p_1^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(w\mathcal{G}_nK_p)\big).$$

Proof. We prove the first point. We have a map $t : p_2^\star\mathcal{M}^{an}_{dR} \to p_1^\star\mathcal{M}^{an}_{dR}$, which is locally represented by wt by proposition 4.81.

Therefore, we have an isomorphism $t^\star : p_1^\star\mathcal{V}_\kappa \to p_2^\star\mathcal{V}_\kappa$ which is locally given by $t^\star f(x'_2 m) = f(x'_1 wtm)$ for trivializations x'_1 and x'_2 of $p_1^\star\mathcal{M}_{dR}$ and $p_2^\star\mathcal{M}_{dR}$. The map $p_2^\star\mathcal{V}_\kappa \to p_1^\star\mathcal{V}_\kappa$ of the lemma is the inverse of the map $t^\star$.

This map is locally isomorphic to the map

$$\begin{array}{rcl} V_\kappa & \to & V_\kappa \\ v & \mapsto & (wt)^{-1}v \end{array}$$

which has eigenvectors of valuation $\langle (wv(t))^{-1}, \nu\rangle$ where ν ranges through the weights of V_κ. Since $t \in T^{d,+}$ and $w \in {}^MW$, it follows that $wv(t) \in X_\star(T)_{\mathbb{Q}}^{M_\mu,+}$ so that $(wv(t))^{-1} \in X_\star(T)_{\mathbb{Q}}^{M_\mu,-}$. The lowest weight of V_κ is $w_{0,M}\kappa$ and therefore, $p^{\langle (wv(t))^{-1}, w_{0,M}\kappa\rangle} V_\kappa^+ \subseteq (wt)^{-1}V_\kappa^+$. We deduce that $p^{\langle (wv(t))^{-1}, w_{0,M}\kappa\rangle} p_2^\star \mathcal{V}_\kappa^+ \subseteq t^\star p_1^\star \mathcal{V}_\kappa^+$ from which we deduce that $p_2^\star\mathcal{V}_\kappa^+ \to p^{\langle wv(t), w_{0,M}\kappa\rangle} p_1^\star \mathcal{V}_\kappa^+$. The proof of the second point is almost identical, it is enough to observe that now $(wv(t))^{-1} \in X_\star(T)_{\mathbb{Q}}^{M_\mu,+}$ and that κ is the highest weight of V_κ. Details are left to the reader. □

Lemma 5.35. *Let $w \in {}^MW$, $\kappa \in X^\star(T^c)$, and $h \in \mathbb{Q}$.*

1. *Let $(\mathcal{U}, \mathcal{Z})$ be a $(+, w, K_{p,m',b})$-allowed support condition. Assume further that $\mathcal{U}$ is a quasi-compact open and that the complement of $\mathcal{Z}$ is quasi-compact. The image of $\mathrm{H}^i_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathcal{V}_\kappa^+)$ in $\mathrm{H}^i_w(K^p, \kappa)^{+,\leq fs}$ is an open and bounded submodule.*
2. *Let $(\mathcal{U}, \mathcal{Z})$ be a $(-, w, K_{p,m',b})$-allowed support condition. Assume further that $\mathcal{U}$ is a quasi-compact open and that the complement of $\mathcal{Z}$ is quasi-compact. The image of $\mathrm{H}^i_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathcal{V}_\kappa^+)$ in $\mathrm{H}^i_w(K^p, \kappa)^{-,\leq fs}$ is an open and bounded submodule.*

Proof. We only treat the first item since the second one follows with minor modifications. We can represent the cohomology $\mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathcal{V}_\kappa)$ by an explicit complex of Banach modules $C^\bullet$. We choose an affinoid covering $\mathfrak{U}_1$ of $\mathcal{U}$ and consider the Čech complex $\check{C}(\mathfrak{U}_1, \mathcal{V}_\kappa)$ which computes $\mathrm{R}\Gamma(\mathcal{U}, \mathcal{V}_\kappa)$. Next we take an affinoid covering $\mathfrak{U}_2$ of $\mathcal{U} \cap \mathcal{Z}^c$ refining the covering $\mathfrak{U}_1 \cap \mathcal{Z}^c$ and consider the Čech complex $\check{C}(\mathfrak{U}_2, \mathcal{V}_\kappa)$ which computes $\mathrm{R}\Gamma(\mathcal{U}\cap\mathcal{Z}^c, \mathcal{V}_\kappa)$. Finally, we represent the cohomology $\mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathcal{V}_\kappa)$ by

$$C^\bullet = \mathrm{Cone}(\check{C}(\mathfrak{U}_1, \mathcal{V}_\kappa) \to \check{C}(\mathfrak{U}_2, \mathcal{V}_\kappa))[-1].$$

We also consider the subcomplex of open and bounded submodules of $C^\bullet$:

$$C^{+,\bullet} = \mathrm{Cone}(\check{C}(\mathfrak{U}_1, \mathcal{V}_\kappa^+) \to \check{C}(\mathfrak{U}_2, \mathcal{V}_\kappa^+))[-1].$$

Any sufficiently regular element T of T^{++} lifts to a compact endomorphism $\tilde{T}$ of $C^\bullet$ and we can consider the direct "slope less or equal than h" factor $C^{\bullet,\leq h}$ of $C^\bullet$. This is a perfect complex of F-vector spaces, whose cohomology groups compute $\mathrm{H}^i_w(K^p,\kappa)^{+,\leq h}$. Denote the projection of $C^{+,\bullet}$ in $C^{\bullet,\leq h}$ by $C^{+,\bullet,\leq h}$. This is a perfect complex of $\mathcal{O}_F$-modules and the image of $\mathrm{H}^i(C^{+,\bullet,\leq h})$ in $\mathrm{H}^i(C^{\bullet,\leq h}) = \mathrm{H}^i_w(K^p,\kappa)^{+,\leq h}$ is therefore an open and bounded submodule. Therefore, for any h, the image of $\mathrm{H}^i(C^{+,\bullet})$ in $\mathrm{H}^i_w(K^p,\kappa)^{+,\leq h}$ is open and bounded. Passing to the limit over h, we deduce that the image of $\mathrm{H}^i(C^{+,\bullet})$ in $\mathrm{H}^i_w(K^p,\kappa)^{+,fs}$ is open and bounded. To prove the lemma, it suffices to show that the map

$$\mathrm{H}^i(C^{+,\bullet}) \to \mathrm{H}^i_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathcal{V}_\kappa^+)$$

has kernel and cokernel of bounded torsion. Using the Čech to cohomology spectral sequence, this follows from lemma 2.30. □

Proof of Theorem 5.33. We only prove the + case. The − case follows with minor modifications. We let $t \in T^+$, and let $T = [K_{p,m',b} t K_{p,m',b}]$. We take $(\mathcal{U}, \mathcal{Z})$ as in lemma 5.35.We find by using lemma 5.34 that we have an endomorphism $p^{-\langle wv(t), w_{0,M}\kappa\rangle} T : \mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathcal{V}_\kappa^+) \to \mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathcal{V}_\kappa^+)$. It follows from lemma 5.35 that $p^{-\langle wv(t), w_{0,M}\kappa\rangle} T$ preserves an open and bounded submodule in $\mathrm{H}^i_w(K^p, \kappa)^{+,fs}$ for all i. For any character of λ of T^+ on $\mathrm{H}^i_w(K^p, \kappa)^{+,fs}$, this implies that $v(\lambda(t)) \geq \langle wv(t), w_{0,M}\kappa\rangle$. The theorem is thus proven. □

5.10. **Comparisons with slope bounds on classical cohomology.**

5.10.1. *Some combinatorics.* Let $\kappa \in X^\star(T^c)^{M_\mu,+}$. We first attach to κ certain subsets of the Weyl group. We let $W(\kappa)^+ = \{w \in W, ww_{0,M}(\kappa+\rho) = w_{0,M}(\kappa+\rho)\}$. We let $W(\kappa)^- = \{w \in W, w(\kappa+\rho) = \kappa+\rho\}$. We let $C(\kappa)^+ = \{w \in W, w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^-_{\mathbb{Q}}\}$ and $C(\kappa)^- = \{w \in W, w^{-1}(\kappa+\rho) \in X^\star(T)^+_{\mathbb{Q}}\}$.

Proposition 5.36.
1. *The set $C(\kappa)^\pm$ is a left principal homogeneous space under $W(\kappa)^\pm$.*
2. *$C(\kappa)^\pm \subseteq {}^M W$.*
3. *$\kappa+\rho$ is regular if and only if $C(\kappa)^\pm$ is reduced to a single element.*
4. *We have $W(\kappa)^+ = w_{0,M} W(\kappa)^- w_{0,M}$ and $C(\kappa)^+ = w_{0,M} C(\kappa)^- w_0$.*
5. *We have $C(\kappa)^\pm = C(-w_{0,M}\kappa - 2\rho_{nc})^\mp$.*

Proof. Left multiplication defines an action of $W(\kappa)^\pm$ on $C(\kappa)^\pm$. Given a weight $\lambda \in X^\star(T)^-$, we have $w\lambda \geq \lambda$ for all $w \in W$. It follows that if $w, w' \in C(\kappa)^\pm$, $w(w')^{-1} \in W(\kappa)^\pm$. The elements of ${}^M W$ are characterized among W by the property that $wX^\star(T)^+ \subseteq X^\star(T)^{M_\mu,+}$. Since $\kappa+\rho$ is M_μ-dominant and regular, the second point follows. The remaining points are evident. □

We now give some more explanations concerning the meaning of these sets and the connection with infinitesimal characters. The element $-w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^+_{\mathbb{Q}}$ is independent of $w \in C(\kappa)^+$, and we denote it by $\nu+\rho$ for $\nu \in X^\star(T)$.

Proposition 5.37 ([Har90a], prop. 3.1.4)**.** *The character $\nu+\rho$ is the dominant representative of the infinitesimal character of the automorphic representations contributing to the cohomology of the sheaves $\mathcal{V}_\kappa$ or $\mathcal{V}_\kappa(-D)$ over $S^{tor}_{K,\Sigma}(\mathbb{C})$.*

Remark 5.38. The infinitesimal character of the automorphic representations contributing to the cohomology of the Serre dual sheaves $\mathcal{V}_{-w_{0,M}\kappa-2\rho_{nc}}$ and $\mathcal{V}_{-w_{0,M}\kappa-2\rho_{nc}}(-D)$ is $-\nu-\rho$. Its dominant representative is therefore $-w_0\nu+\rho$.

It is important to record the formulas that allow us to switch between the infinitesimal character and the weight:

$$\begin{aligned} \nu &= -w^{-1}w_{0,M}(\kappa+\rho) - \rho, \ \forall w \in C(\kappa)^+ \\ \kappa &= -w_{0,M}w(\nu+\rho) - \rho, \ \forall w \in C(\kappa)^+ \\ \nu &= -w_0 w^{-1}(\kappa+\rho) - \rho, \ \forall w \in C(\kappa)^- \\ \kappa &= -ww_0(\nu+\rho) - \rho, \ \forall w \in C(\kappa)^-. \end{aligned}$$

We also introduce the notation $\ell_{\min}(\kappa) = \min_{w\in C(\kappa)^+} \ell_+(w) = \min_{w\in C(\kappa)^-} \ell_-(w)$ and $\ell_{\max}(\kappa) = \max_{w\in C(\kappa)^+} \ell_+(w) = \max_{w\in C(\kappa)^-} \ell_-(w)$. Here the equalities follow from the fact that for $w \in {}^M W$, $\ell_+(w_{0,M}ww_0) = d - \ell_+(w) = \ell_-(w)$. Moreover $\ell_{\min}(\kappa) = \ell_{\max}(\kappa)$ if and only if $\kappa+\rho$ is regular. We note that we expect the automorphic vector bundle $\mathcal{V}_\kappa$ to have interesting cohomology in the range $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$.

Theorem 5.39. *Let π be an automorphic representation contributing to the cohomology of the sheaves $\mathcal{V}_\kappa$ or $\mathcal{V}_\kappa(-D)$ over $S^{tor}_{K,\Sigma}(\mathbb{C})$. Assume that π_∞ is (essentially) tempered. Then π can contribute to the cohomology in the range $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$.*

Proof. This follows from a combination of results of Blasius-Harris-Ramakrishnan, Mirkovich, Schmid and Williams. See [Har90a], thm. 3.4 and thm. 3.5. □

5.10.2. *Slope bounds on classical coherent cohomology.* In light of the spectral sequences of section 5.5, conjecture 5.29 suggests the following conjectural slope bound for classical cohomology:

Conjecture 5.40. *Let $\kappa \in X^\star(T)^{M_\mu,+}$ and let $\chi : T(\mathbb{Z}_p) \to F^\times$ be a finite order character. Let $\nu = -w^{-1}w_{0,M}(\kappa+\rho) - \rho$ for any $w \in C(\kappa)^+$. For any eigensystem $\lambda : T^\pm \to \overline{F}^\times$ occurring in the classical cohomologies $\mathrm{R}\Gamma(K^p, \kappa, \chi)^{\pm, fs}$ or $\mathrm{R}\Gamma(K^p, \kappa, \chi, cusp)^{\pm, fs}$, we have:*

1. *In the $+$ case, $v(\lambda) \geq -\nu$.*
2. *In the $-$ case, $v(\lambda) \leq -w_0\nu$.*

Remark 5.41. The $+$ and $-$ statements are in fact equivalent, in view of the discussion of section 4.3 and in particular the isomorphism between the Jacquet modules for U and $\overline{U}$ given by w_0.

Proposition 5.42. *Conjecture 5.29 implies conjecture 5.10.2.*

Proof. We treat the non cuspidal $+$ case, the others are identical. By the spectral sequence of section 5.5 λ occurs in $\mathrm{R}\Gamma_{w'}(K^p, \kappa, \chi)^{+, fs}$ for some $w' \in {}^MW$, and hence by conjecture 5.29 we have

$$s(\lambda) \geq {w'}^{-1}w_{0,M}(\kappa+\rho) + \rho = -({w'}^{-1}w)\cdot\nu \geq -\nu$$

where the last inequality follows from lemma 5.55 below, using that $\nu+\rho \in X^\star(T)^+_{\mathbb{R}}$. □

We are not able to prove completely, however we will see that it holds when $\kappa+\rho$ is regular in theorem 5.44 in the next section, and we will eventually use p-adic interpolation to prove it for interior cohomology in theorem 6.48, and so in particular it holds for compact Shimura varieties.

We now explain the relation of this conjecture with other known and conjectured slope bounds on classical cohomology.

5.10.3. *Slope bounds on Betti cohomology.* Let $\nu \in X^\star(T)^+$. Let W_ν be the corresponding irreducible representation of G with highest weight ν defined over F. Over $S_K(\mathbb{C})$, we can construct a local system $\mathcal{W}_\nu^\vee$ attached to $W_\nu^\vee$ and we can consider the Betti cohomology groups $\mathrm{H}^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$ and $\mathrm{H}^\star_c(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$.

Proposition 5.43. *Assume that $K = K^pK_p$ with $K_p = K_{p,m,b}$. Let $\nu \in X^\star(T)^+$. For any eigensystem $\lambda : T^\pm \to \overline{F}^\times$ for the action of $\mathcal{H}^\pm_{p,m,b}$ on $\mathrm{H}^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, fs}$ or $\mathrm{H}^\star_c(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, fs}$, we have:*

1. *$v(\lambda) \geq -\nu$ in the $+$ case,*
2. *$v(\lambda) \leq -w_0\nu$ in the $-$ case.*

Proof. This is a straightforward adaptation of [Laf11], prop. 3.1 which considers the case where K_p is hyperspecial in an unramified group G. Recall that G splits over F. The representation $W_\nu^\vee$ admits a lattice $W_\nu^{\vee,+}$ which is stable under the action of $G(\mathcal{O}_F)$ and admits a weight decomposition with respect to the action of T. For any $t \in T^+$, we have that $t(W_\nu^{\vee,+}) \subseteq (-\nu)(t)W_\nu^{\vee,+}$ and for any $t \in T^-$, we have that $t(W_\nu^{+,\vee}) \subseteq (-w_0\nu)(t)W_\nu^{+,\vee}$, as $-\nu$ and $-w_0\nu$ are the lowest and highest weights of $W_\nu^\vee$. The lattice $W_\nu^{+,\vee}$ gives an $\mathcal{O}_F$-local system $\mathcal{W}_\nu^{+,\vee}$ such that $\mathcal{W}_\nu^{+,\vee} \otimes_{\mathcal{O}_F} F = \mathcal{W}_\nu^\vee$. The image of $\mathrm{H}^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^{+,\vee})$ in $\mathrm{H}^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$ (resp. of $\mathrm{H}_c^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^{+,\vee})$ in $\mathrm{H}_c^\star(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$) is a lattice L (resp. L_c). For any $t \in T^+$, we find that $[K_ptK_p](L) \subseteq (-\nu)(t)L$ and $[K_ptK_p](L_c) \subseteq (-\nu)(t)L_c$. For any $t \in T^-$, we find that $[K_ptK_p](L) \subseteq (-w_0\nu)(t)L$ and $[K_ptK_p](L_c) \subseteq (-w_0\nu)(t)L_c$. □

Using this proposition, we can prove conjecture 5.10.2 when the weight is regular.

Corollary 5.44. *Conjecture 5.10.2 holds when $\kappa + \rho$ is G-regular.*

Proof. Since $\kappa + \rho$ is G regular, there is a unique $\nu \in X^\star(T)^+$ and a unique $v \in W$ such that $-\kappa - \rho = v(\nu + \rho)$. By the definition we have $C(\kappa)^+ = \{w_{0,M}v\}$ and $C(\kappa)^- = \{vw_0\}$.

By the degeneration of Faltings's dual BGG spectral sequences (see for example [Har90a] section 4, [HZ01] Cor. 4.2.3), $\bigoplus_{w' \in {}^MW} \mathrm{H}^{i-l(w')}(K^p, -w'w_0(\nu + \rho) - \rho, \chi)^{\pm, fs}$ embeds Hecke-equivariantly in $\mathrm{H}^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$ and $\bigoplus_{w' \in {}^MW} \mathrm{H}^{i-l(w')}(K^p, -w'w_0(\nu + \rho) - \rho, \chi, cusp)^{\pm, fs}$ embeds Hecke-equivariantly in $\mathrm{H}_c^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)$. The estimate follows from proposition 5.43. □

5.10.4. *Connection with* [FP19]. We also want to explain that conjecture 5.29 is also compatible with conjecture 4.5 of [FP19] at spherical level (inspired by [Laf11]) which is a translation of the Katz-Mazur inequality on the cohomology of algebraic varieties to the automorphic setting.

Let Γ be the Galois group of $F/\mathbb{Q}_p$, acting on $X^\star(T)$. The projection $X^\star(T)_\mathbb{R} \to X^\star(T^d)_\mathbb{R}$ induces an isomorphism $X^\star(T)_\mathbb{R}^\Gamma \to X^\star(T^d)_\mathbb{R}$, an inverse is given by $\lambda \mapsto \frac{1}{|\Gamma|}\sum_{\sigma \in \Gamma} \sigma.\tilde{\lambda}$ for $\lambda \in X^\star(T^d)_\mathbb{R}$ and $\tilde{\lambda}$ being any lift of λ to $X^\star(T)_\mathbb{R}$.

We can therefore identify $X^\star(T^d)_\mathbb{R}$ as a subspace of $X^\star(T)_\mathbb{R}$ and the partial order on $X^\star(T^d)_\mathbb{R}$ is the one induced by the partial order on $X^\star(T)_\mathbb{R}$. This is the point of view adopted in [FP19].

We will assume that $G_{\mathbb{Q}_p}$ is of the form $\mathrm{Res}_{L/\mathbb{Q}_p} G_0$ where L is a finite extension of $\mathbb{Q}_p$ and G_0 is an unramified reductive group over L. In [FP19], the group $G_{\mathbb{Q}_p}$ was assumed to be unramified, but the same conjecture can be made in this level of generality, and is interesting for applications. We assume that $K_p \subseteq G(\mathbb{Q}_p) = G_0(L)$ is a hyperspecial subgroup of $G_0(L)$ We consider the classical cohomology $\mathrm{R}\Gamma(S^{tor}_{K_pK^p,\Sigma}, \mathcal{V}_\kappa)$ or $\mathrm{R}\Gamma(S^{tor}_{K_pK^p,\Sigma}, \mathcal{V}_\kappa(-D))$. We let $\infty(\kappa) \in X^\star(T)^+$ be the dominant representative of $-\kappa - \rho$ which is the infinitesimal character of automorphic representations contributing to $\mathrm{R}\Gamma(S^{tor}_{K_pK^p,\Sigma}, \mathcal{V}_\kappa)$ or $\mathrm{R}\Gamma(S^{tor}_{K_pK^p,\Sigma}, \mathcal{V}_\kappa(-D))$. Let $\mathcal{H}(G(\mathbb{Q}_p), K_p)$ be the spherical Hecke algebra. Recall that T^d is the maximal split subtorus of T. Let T_0 be a maximal torus of G_0. We can assume that $T = \mathrm{Res}_{L/\mathbb{Q}_p} T_0$. Let T_0^d be the maximal split sub-torus of T_0. Then T_0^d is naturally defined over $\mathbb{Q}_p$ and the diagonal map $T_0^d \hookrightarrow \mathrm{Res}_{L/\mathbb{Q}_p} T_0^d \hookrightarrow T$ indentifies T_0^d and T^d. We let W_0^d be the sub-group of the (geometric) Weyl group of G_0 which stabilizes T_0^d. We fix an element $p^{\frac{1}{2}} \in F$ (this is always possible if we enlarge F).

We have the Satake isomorphism:

$$\mathcal{S} : \mathcal{H}(G(\mathbb{Q}_p), K_p) \otimes_{\mathbb{Z}} F \to F[X_\star(T_0^d)]^{W_0^d}$$

and to any $\lambda : \mathcal{H}(G(\mathbb{Q}_p), K_p) \to \overline{F}$, we can attach an semi-simple σ-conjugacy class $c \in (X^\star(T_0^d) \otimes \bar{F}^\times)/W_0^d = {}^L G_0(\bar{F})^{ss}/\sigma - \mathrm{conj}$, where ${}^L G_0(\bar{F}) = \hat{G}_0 \rtimes \mathbb{Z}$ is the Langlands group of G_0. This is the semi-direct product of the dual group $\hat{G}_0$ by the free group $\mathbb{Z}$ generated by σ. The action of σ on G_0^D is the one induced by the Frobenius which is a generator of the Galois group of the unramified extension of L which splits G_0. In particular, if G_0 is split, this action is trivial.

Recall the valuation map $v : \bar{F}^\times \to \mathbb{R}$. Applying the valuation to c gives the element $\mathrm{Newt}_v(c) \in X^\star(T_0^d)_{\mathbb{R}}/W_0^d = X^\star(T_0)_{\mathbb{R}}^\Gamma/W_0^d = X^\star(T_0^d)_{\mathbb{R}}^+ = (X^\star(T_0)_{\mathbb{R}}^+)^\Gamma$.

We also remark that $T(\mathbb{Q}_p)/T(\mathbb{Z}_p) = T_0(L)/T_0(\mathcal{O}_L) = X_\star(T_0^d)$ via the map $\lambda \in X_\star(T_0^d) \mapsto \lambda(\varpi_L)$ where ϖ_L is a uniformizing element in L. We have also defined a valuation map $v : T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \to X_\star(T^d) \otimes \mathbb{Q}$. We therefore get a map $v : X_\star(T_0^d) \to X_\star(T^d) \otimes \mathbb{Q}$. This map is given by multiplication by $v(\varpi_L)$ (via the identification $X_\star(T_0^d) = X_\star(T^d)$).

We have the following conjecture (which is [FP19], conj. 4.5 in the unramified case):

Conjecture 5.45. *For any $t \in X_\star(T_0^d)_{\mathbb{R}}^+$, we have*

$$\langle t, \mathrm{Newt}_v(c) \rangle \leq \langle v(t), -w_0 \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \sigma.\infty(\kappa) \rangle.$$

Remark 5.46. If L is unramified, $X_\star(T_0^d)$ is canonically identified with $X_\star(T^d)$ via the valuation map v, and the above identity simply writes: $\mathrm{Newt}_v(c) \leq -w_0 \frac{1}{|\Gamma|} \sum_{\sigma\in\Gamma} \sigma.\infty(\kappa)$ in $X_\star(T)$.

We can reformulate this conjecture in another way. Any element $t \in X_\star(T_0^d)^+$ gives a σ-equivariant representation of $\hat{G}_0$ (of highest weight t) and therefore a representation of ${}^L G_0$.

Lemma 5.47. *The conjecture 5.45 holds if and only if, for any $t \in X_\star(T_0^d)^+$ viewed as a dominant character of the Langlands group, and with associated highest weight representation V_t, we have that for any eigenvalue x of c on V_t,*

$$v(x) \geq -\langle v(t), \infty(\kappa) \rangle.$$

Remark 5.48. We remark that $-\infty(\kappa)$ is the anti-dominant representative of $\kappa + \rho$.

Proof. It follows from lemma 3.6 of [FP19] that the conjecture is equivalent to the statement that $v(\mathrm{Tr}(c|V_t)) \geq \langle w_0(v(t)), -w_0\infty(\kappa)\rangle = \langle v(t), -\infty(\kappa)\rangle$. Therefore, the converse implication holds. Let us prove the direct implication. If there is a unique eigenvalue of c on V_t with minimal valuation, we deduce that for any eigenvalue x of c on V_t, $v(x) \geq v(\mathrm{Tr}(c|V_t))$. Otherwise, let $x_1, \cdots, x_i$ be the i-th eigenvalues of minimal valuation. Then one considers the representation $\Lambda^i V_t$ and we find $iv(x_1) = v(\mathrm{Tr}(c \rtimes \sigma|\Lambda^i V_t)) \geq i\langle v(t), -\infty(\kappa)\rangle$. □

Before we state our main compatibility, we need to recall certain relations between the spherical and Iwahori Hecke algebras. We have the spherical Hecke algebra $\mathcal{H}(G, K_p)$ and the Iwahori Hecke algebra $\mathcal{H}(G, K_{p,1,0})$. These are algebras for the convolution product for a Haar measure normalized by $\mathrm{vol}(K_p) = 1$ (respectively $\mathrm{vol}(K_{p,1,0}) = 1$). We have also introduced a subalgebra $\mathcal{H}^+_{p,1,0}$ of

$\mathcal{H}(G, K_{p,1,0})$, isomorphic to $\mathbb{Z}[T^+]$, and generated by the elements $[K_{p,1,0}tK_{p,1,0}]$ with $t \in T^+$.

We now consider the twisted embedding

$$F[X_\star(T_0^d)^+] \hookrightarrow \mathcal{H}(G, K_{p,1,0}) \otimes_{\mathbb{Z}} F$$

which sends $t \in T^+/T(\mathbb{Z}_p) = X_\star(T_0^d)^+$ to $q^{-\langle t,\rho_0\rangle}[K_{p,1,0}tK_{p,1,0}]$ where q is the cardinal of $\mathcal{O}_L/\varpi_L$ and ρ_0 is half the sum of the positive roots in G_0. All the operators $q^{-\langle t,\rho\rangle}[K_{p,1,0}tK_{p,1,0}]$ are invertible in $\mathcal{H}(G, K_{p,1,0}) \otimes_{\mathbb{Z}} F$ and this map extends to an embedding

$$F[X_\star(T_0^d)] \hookrightarrow \mathcal{H}(G, K_{p,1,0}) \otimes_{\mathbb{Z}} F.$$

Moreover, $F[X_\star(T_0^d)]^{W_d}$ is the center of $\mathcal{H}(G, K_{p,1,0}) \otimes_{\mathbb{Z}} F$. Let $e_{K_p} \in \mathcal{H}(G, K_{p,1,0})$ be the idempotent equal to characteristic function of K_p divided by the volume of K_p. The natural isomorphism:

$$\mathcal{H}(G, K_p) \otimes F \to e_{K_p}(\mathcal{H}(G, K_{p,1,0}) \otimes F)e_{K_p}$$

induces an isomorphism

$$\mathcal{H}(G, K_p) \otimes F \to e_{K_p}F[X_\star(T_0^d)]^{W_d}$$

which is the Satake isomorphism.

Corollary 5.49. *Let π be an irreducible smooth admissible representation of $G(\mathbb{Q}_p)$ defined over $\bar{F}$. Assume that $\pi^{K_p} \neq 0$. Then π^{K_p} is one dimensional. Let $c \in (X^\star(T_0^d) \otimes \bar{F}^\times)/W_0^d$ be the semi-simple σ-conjugacy class corresponding to the action of $\mathcal{H}(G, K_p)$ on π^{K_p}. Any eigensystem of $F[X_\star(T_0^d)]$ acting on $\pi^{K_{p,1,0}}$ is given by a lift $\tilde{c} \in X^\star(T_0^d) \otimes \bar{F}^\times$ of c.*

Remark 5.50. In particular for any $t \in T^+/T(\mathbb{Z}_p) = X_\star(T_0^d)^+$, the eigenvalues of $q^{-\langle t,\rho_0\rangle}[K_{p,1,0}tK_{p,1,0}]$ acting on $\pi^{K_{p,1,0}}$ are among the eigenvalues of c acting on the representation V_t of the Langlands dual group.

Definition 5.51. *We say that a class in $\mathrm{H}^i_w(K_p, \kappa)^{+,fs}$ or $\mathrm{H}^i_w(K_p, \kappa, cusp)^{+,fs}$ is classical if maps to a non zero class of the abutment of the spectral sequence of theorem 5.15.*

Let f be a classical eigenclass for the action of $\mathcal{H}^+_{p,1,0}$ on $\mathrm{H}^i_w(K_p, \kappa)^{+,fs}$ or $\mathrm{H}^i_w(K_p, \kappa, cusp)^{+,fs}$. Then f gives an eigenclass f_{cl} in a subquotient of $\mathrm{H}^i(\mathcal{S}^{tor}_{K_{p,1,0}K^p,\Sigma}, \mathcal{V}_\kappa)$ or $\mathrm{H}^i(\mathcal{S}^{tor}_{K_{p,1,0}K^p,\Sigma}, \mathcal{V}_\kappa(-D))$. These classical cohomologies are $\mathcal{H}(G, K_{p,1,0})$-modules and admit Jordan-Holder filtrations, whose graded pieces are of the form $\pi^{K_{p,1,0}}$ for smooth irreducible admissible $G(\mathbb{Q}_p)$-representations π defined over $\bar{F}$. We say that the classical eigenclass f is spherical if f_{cl} belongs to a π which admits spherical vectors. We let $f_{cl,sph}$ be a corresponding spherical vector.

Proposition 5.52. *Let $w \in C(\kappa)^+$. Let f be an eigenclass for the action of $\mathcal{H}^+_{p,1,0}$ on $\mathrm{H}^i_w(K_p, \kappa)^{+,fs}$ or $\mathrm{H}^i_w(K_p, \kappa, cusp)^{+,fs}$. Assume that f is classical spherical with associated spherical class $f_{cl,sph}$. If conjecture 4.5 of* [FP19] *holds for the eigensystem of $f_{cl,sph}$ then conjecture 5.29 holds for the eigensystem of f.*

Proof. Let c be the semi-simple conjugacy class arising from the spherical eigenclass. Let λ be the character of $\mathcal{H}_{p,1,0}$. For each $t_0 \in T^+/T(\mathbb{Z}_p) = X_\star(T_0^d)^+$, we see that $\lambda(t_0)q^{-\langle t_0,\rho_0\rangle}$ is an eigenvalue for c acting on V_{t_0}. It follows from lemma 5.47 that $v(\lambda(t_0)) - v(q)\langle t_0, \rho_0\rangle \geq \langle v(t_0), -\infty(\kappa)\rangle$. Where via the valuation map

$v : X_\star(T_0^d) \to X_\star(T^d) \otimes \mathbb{Q}$, t_0 maps to $v(\varpi_L)t$, where we let t be the element corresponding to t_0 via the isomorphism $X_\star(T_0^d) = X_\star(T^d)$. So this identity can be re-written:

$$v(\lambda(t))v(\varpi_L) - v(q)\langle t_0, \rho_0\rangle \geq v(\varpi_L)\langle t, -\infty(\kappa)\rangle.$$

It remains to remark that $v(q)v(\varpi_L)^{-1} = [L : \mathbb{Q}_p]$ and that $\langle t_0, \rho_0\rangle[L : \mathbb{Q}_p] = \langle t, \rho\rangle$. □

5.11. **Slopes and small slope conditions.** We will now define several "small slope" conditions that will occur in this paper. The reason that there are so many conditions is explained as follows. First we consider small slope condition on the coherent cohomology in weight κ, but also conditions on the Betti cohomology in weight $-w_0\nu$. Second, the conditions needed to obtain a vanishing theorem are not exactly the same as those needed to obtain classicity theorems. For example, on a Shimura set the vanishing theorem is trivial and does not require any slope condition, but when one considers the theory of p-adic algebraic automorphic forms, there is a slope condition to achieve classicity. Finally, in our setting there are two types of control theorems. We have control theorems for cohomologies of classical automorphic sheaves, but also control theorems for cohomologies valued in Banach sheaves. All this explains the large variety of slope conditions we need. We begin with the small slope condition and then turn to the strongly small slope condition which we need to use because we were not able to prove conjecture 5.29.

5.11.1. *Small slope conditions.*

Definition 5.53. *Let $\lambda \in X^\star(T^d)_\mathbb{R}$.*

- *Let $\nu \in X^\star(T)$ satisfy $\nu + \rho \in X^\star(T)^+_\mathbb{R}$.*
  - *We say λ satisfies $+, ss(\nu)$ if for all $w \in W$ with $w \cdot \nu \neq \nu$,*
    $$\lambda \not\geq -w \cdot \nu.$$
  - *We say λ satisfies $-, ss(\nu)$ if for all $w \in W$ with $w \cdot \nu \neq \nu$,*
    $$\lambda \not\leq -w \cdot (w_0\nu).$$
- *Let $\kappa \in X^\star(T)^{M,+}$.*
  - *We say λ satisfies $+, ss^M(\kappa)$ if for all $w \in {}^MW \setminus C(\kappa)^+$,*
    $$\lambda \not\geq w^{-1}w_{0,M}(\kappa + \rho) + \rho.$$
  - *We say λ satisfies $-, ss^M(\kappa)$ if for all $w \in {}^MW \setminus C(\kappa)^-$,*
    $$\lambda \not\leq w^{-1}(\kappa + \rho) - \rho.$$
- *Let $\kappa \in X^\star(T)$ and let $w \in {}^MW$.*
  - *We say λ satisfies $+, ss_{M,w}(\kappa)$ if for all $w' \in W_M$, $w' \neq 1$,*
    $$\lambda \not\geq w^{-1}w_{0,M}w'(\kappa + \rho) + \rho.$$
  - *We say λ satisfies $-, ss_{M,w}(\kappa)$ if for all $w' \in W_M$, $w' \neq 1$,*
    $$\lambda \not\leq w^{-1}w'(\kappa + \rho) - \rho.$$

To orient the reader, we give a brief summary of how these conditions arise:

- The condition $\pm, ss^M(\kappa)$ will appear in the "geometric" classicality theorem relating classical cohomology and overconvergent cohomology, as well as vanishing theorems for classical cohomology in section 5.12.

- The condition $\pm, ss_{M,w}(\kappa)$ will arise in the second classicality theorem "at the level of the sheaf" relating overconvergent and locally analytic cohomologies in algebraic weights in section 6.6.
- The condition $\pm, ss(\nu)$ is the usual small slope condition that arises in works on p-adic modular forms from the Betti perspective. We shall see in proposition 5.61 below that it is the combination of the other two conditions.

We first observe that the $+$ and $-$ conditions are related by two symmetries:

Proposition 5.54. *Let $\lambda \in X^\star(T^d)_{\mathbb{R}}$.*

(1) *Let $\nu \in X^\star(T)$ satisfy $\nu + \rho \in X^\star(T)^+_{\mathbb{R}}$. Then the following are equivalent:*
   - *λ satisfies $-, ss(\nu)$.*
   - *$w_0(\lambda)$ satisfies $+, ss(\nu)$.*
   - *$-\lambda$ satisfies $+, ss(-w_0\nu)$.*

(2) *Let $\kappa \in X^\star(T)^{M,+}$. Then the following are equivalent:*
   - *λ satisfies $-, ss^M(\kappa)$.*
   - *$w_0(\lambda)$ satisfies $+, ss^M(\kappa)$.*
   - *$-\lambda$ satisfies $+, ss^M(-w_{0,M}\kappa - 2\rho_{nc})$.*

(3) *Let $\kappa \in X^\star(T)^{M,+}$ and let $w \in {}^MW$. Then the following are equivalent:*
   - *λ satisfies $-, ss_{M,w}(\kappa)$.*
   - *$w_0(\lambda)$ satisfies $+, ss_{M,w_{0,M}ww_0}(\kappa)$.*
   - *$-\lambda$ satisfies $+, ss_{M,w}(-w_{0,M}\kappa - 2\rho_{nc})$.*

The first symmetry is related to the fact that when we have a smooth, admissible representation of $G(\mathbb{Q}_p)$, the action of w_0 exchanges the $+$ and $-$ finite slope parts (see section 5.12.3 below.) The second symmetry is related to Poincare and Serre duality.

Now we try to further explain the meaning of these small slope conditions and make them more explicit. In view of the symmetries above we only consider the $+$ case. For $\nu \in X^\star(T)$ we introduce the notation $W_\nu = \{w \in W \mid w \cdot \nu = \nu\}$.

We will use the following standard lemma.

Lemma 5.55. *Let $\nu \in X^\star(T)^+_{\mathbb{R}} - \rho$ and let $w, w' \in W$. If $w \leq w'$ then $w' \cdot \nu \preceq w \cdot \nu$.*

Proof. By the definition of the Bruhat order and induction, it suffices to treat the case that $w' = s_\alpha w$ with $\alpha \in \Phi^+$ with $l(w') > l(w)$, which implies that $w^{-1}\alpha \in \Phi^+$ by [Hum90] 5.7. Then $w' \cdot \nu = w \cdot \nu - \langle \alpha^\vee, w(\nu + \rho)\rangle\alpha$, and $\langle \alpha^\vee, w(\nu+\rho)\rangle = \langle (w^{-1}\alpha)^\vee, \nu + \rho\rangle \geq 0$. □

We now give some alternative characterizations of the condition $+, ss(\nu)$.

Proposition 5.56. *The following conditions on $\lambda \in X^\star(T^d)_{\mathbb{R}}$ are equivalent:*

(1) *$\lambda \not\geq -w \cdot \nu$ for all $w \in W \setminus W_\nu$, i.e. λ satisfies $+, ss(\nu)$.*
(2) *$\lambda \not\geq -s_\alpha \cdot \nu$ for all $\alpha \in \Delta$ with $s_\alpha \notin W_\nu$.*

Moreover if we additionally assume that $\lambda \geq -\nu$ then we have the further equivalent condition:

(3) *$\lambda = -\nu + \sum_{\alpha \in \Delta_d} c_\alpha \alpha$ with*

$$c_\alpha < \min_{\beta \in r^{-1}(\alpha), s_\beta \notin W_\nu} \langle \beta^\vee, \nu\rangle + 1.$$

Proof. Clearly the first condition implies the second. For the converse, given $w \in W \setminus W_\nu$, we have $w \geq s_\alpha$ for some $\alpha \in \Delta$, $s_\alpha \notin W_\nu$ (write w as a reduced product of simple reflections, not all the factors can fix ν as w doesn't.) Then $-w \cdot \nu \geq -s_\alpha \cdot \nu$ by lemma 5.55 and so $\lambda \not\geq -s_\alpha \cdot \nu$ implies $\lambda \not\geq -w \cdot \nu$.

Under the hypothesis $\lambda \geq -\nu$, the equivalence of the second and third conditions is immediate from the formula $-s_\beta \cdot \nu = -\nu + (\langle \beta^\vee, \nu \rangle + 1)\beta$. □

Now we consider the condition $+, ss^M(\kappa)$. For $\kappa \in X^\star(T)$ we can write $-\kappa - \rho = v(\nu + \rho)$ for a unique $\nu \in X^\star(T)$ with $\nu + \rho \in X^\star(T)^+_{\mathbb{R}}$, and $v \in W$ uniquely determined up to right multiplication by W_ν.

Proposition 5.57. *Let $\kappa \in X^\star(T)^{M,+}$. Then with ν and v as above, the following conditions on $\lambda \in X^\star(T^d)_{\mathbb{R}}$ are equivalent:*

(1) *$\lambda \not\geq w^{-1} w_{0,M}(\kappa + \rho) + \rho$ for all $w \in {}^M W \setminus C(\kappa)^+$, i.e. λ satisfies $+, ss^M(\kappa)$.*
(2) *$\lambda \not\geq -w \cdot \nu$ for all $w \in ({}^M W)^{-1} \cdot C(\kappa)^+ \setminus W_\nu$.*
(3) *$\lambda \not\geq -s_\alpha \cdot \nu$ for all $\alpha \in \Delta$ with $s_\alpha \in ({}^M W)^{-1} \cdot C(\kappa)^+ \setminus W_\nu$.*

Moreover if we additionally assume that $\lambda \geq -\nu$ then we have the further equivalent condition:

(4) *$\lambda = -\nu + \sum_{\alpha \in \Delta_d} c_\alpha \alpha$ with*

$$c_\alpha < \min_{\beta \in r^{-1}(\alpha), s_\beta \in ({}^M W)^{-1} \cdot C(\kappa)^+ \setminus W_\nu} \langle \beta^\vee, \nu \rangle + 1.$$

Proof. The second condition is a direct translation of the first: we have $C(\kappa)^+ = w_{0,M} v W_\nu$, and we can write

$$w^{-1} w_{0,M}(\kappa + \rho) + \rho = (w^{-1} w_{0,M} v) v^{-1}(\kappa + \rho) + \rho = -(w^{-1} w_{0,M} v) \cdot \nu$$

and so the first condition is equivalent to $\lambda \not\geq -w \cdot \nu$ for $w \in ({}^M W)^{-1} w_{0,M} v \setminus W_\nu$, which is equivalent to condition 2 because $({}^M W)^{-1} w_{0,M} v W_\nu = ({}^M W)^{-1} \cdot W_\nu$.

The second condition clearly implies the third. For the converse, to argue as in the proof of proposition 5.56 we need to show that for all $w \in W$ with $w \in ({}^M W)^{-1} \cdot C(\kappa)^+ \setminus W_\nu$, we have $w \geq s_\alpha$ with $\alpha \in \Delta$ and $s_\alpha \in ({}^M W)^{-1} \cdot C(\kappa)^+ \setminus W_\nu$. To do this, suppose $w = (w')^{-1} w''$ with $w' \in {}^M W$, $w'' \in C(\kappa)^+$, and let $w = s_1 \cdots s_n$ be a reduced expression as a product of simple reflections. Choose k such that $s_k \notin W_\nu$ but $s_{k+1}, \ldots, s_n \in W_\nu$ (such a k exists as $w \notin W_\nu$.) Then

$$s_k = (w' s_1 \cdots s_{k-1})^{-1} (w'' s_{k+1} \cdots s_n) \in ({}^M W)^{-1} \cdot C(\kappa)^+$$

using lemma 5.58 below to see that $w' s_1 \cdots s_{k-1} \in {}^M W$. Moreover $s_k \leq w$, and $s_k \notin W_\nu$, so we are done.

The equivalence of the third and fourth conditions is exactly as in proposition 5.56. □

Lemma 5.58. *Let $w, w' \in {}^M W$ and let $w^{-1} w' = s_1 \cdots s_n$ be a reduced expression as a product of simple roots. Then $w s_1 \cdots s_i \in {}^M W$ for all $1 \leq i \leq n$.*

Proof. We begin with the following claim: if $\alpha \in \Phi^+$, $\beta \in \Delta$, and $u \in W$ are such that $l(u s_\beta) > l(u)$ and $u^{-1}\alpha \in \Phi^-$, then $(u s_\beta)^{-1}\alpha \in \Phi^-$. Indeed, $(u s_\beta)^{-1}\alpha = s_\beta(u^{-1}\alpha) \in \Phi^+$ would imply $u^{-1}(\alpha) = -\beta$, and hence $u\beta \in \Phi^-$, contradicting $l(u s_\beta) > l(u)$.

Applying the claim inductively we see that if $\alpha \in \Phi^+$, and $(s_1 \cdots s_i)^{-1}\alpha \in \Phi^-$, then $(s_1 \cdots s_n)^{-1}\alpha \in \Phi^-$.

Now if $ws_1\cdots s_i \notin {}^MW$, there exists $\beta\in\Delta_M$ with $(ws_1\cdots s_i)^{-1}\beta = (s_1\cdots s_i)^{-1}(w^{-1}\beta)\in \Phi^-$. But $w\in {}^MW$ implies $w^{-1}\beta\in\Phi^+$, and so from the above with $\alpha = w^{-1}\beta$ we deduce $(s_1\cdots s_n)^{-1}(w^{-1}(\beta)) = {w'}^{-1}(\beta)\in\Phi^-$, and hence $w'\notin {}^MW$. □

Now we turn to the conditions $+, ss_{M,w}(\kappa)$. Let $\kappa\in X^\star(T)^+$ and let $w\in {}^MW$. We let $v = w_{0,M}w$ and we let ν be defined by the formula $v(\nu+\rho) = -\kappa-\rho$. Note that we have $\nu+\rho\in X^\star(T)^+_{\mathbb{R}}$ if and only if $w\in C(\kappa)^+$, which we have not assumed for the moment.

Proposition 5.59. *Let $\kappa\in X^\star(T)^{M,+}$ and let $w\in W$. Let ν and v be as above. The following conditions on $\lambda\in X^\star(T^d)_{\mathbb{R}}$ are equivalent:*

1. *$\lambda \not\geq w^{-1}w_{0,M}w'(\kappa+\rho)+\rho$ for all $w'\in W_M$, $w'\neq 1$, i.e. λ satisfies $+, ss_{M,w}(\kappa)$*
2. *$\lambda\not\geq -(v^{-1}w'v)\cdot\nu$ for all $w'\in W_M$, $w'\neq 1$.*
3. *$\lambda\not\geq -(v^{-1}s_\alpha v)\cdot\nu$ for all $\alpha\in\Delta_M$.*

Proof. The equivalence of the first and second conditions is a direct translation.

For the equivalence of the second and third points, we introduce temporarily the notation $\lambda_1\preceq_{M,w}\lambda_2$ if $\lambda_2-\lambda_1\in\mathbb{R}_{\geq 0}w^{-1}\Delta_M$ for $\lambda_1,\lambda_2\in X^\star(T)_{\mathbb{R}}$. Since $w\in {}^MW$ we have $w^{-1}\Delta_M\subseteq\Phi^+$ and hence $\lambda_1\preceq_{M,w}\lambda_2$ implies $\lambda_1\preceq\lambda_2$ and hence $\lambda_1\leq\lambda_2$.

Now applying lemma 5.55 for the group M, we see that for $\nu'\in X^\star(T)^{-,M}_{\mathbb{R}}-\rho_M$ and $w',w''\in W_M$ with $w'\leq w''$ we have $w'\cdot\nu'\preceq_{M,1} w''\cdot\nu'$, hence $(w_{0,M}w'')\cdot\nu'\preceq_{M,1}(w_{0,M}w'')\cdot\nu'$, and hence $(w^{-1}w_{0,M}w'')\cdot\nu'\preceq_{M,w}(w^{-1}w_{0,M}w')\cdot\nu'$. We apply this with $\nu' = v\cdot\nu = -\kappa-\rho\in X^\star(T)^{-,M}_{\mathbb{R}}-\rho_M$ to deduce that for $w'\leq w''$ we have $-(v^{-1}w''v)\cdot\nu\geq -(v^{-1}w'v)\cdot\nu$, and hence the third condition implies the second. □

Remark 5.60. Note that for $w\in W_M$, $(v^{-1}wv)\cdot\nu = \nu$ implies $w=1$. Indeed, then $w\cdot(-\kappa-2\rho) = -\kappa-2\rho$ or equivalently $w(-\kappa-\rho_M) = -\kappa-\rho_M$, and hence that $w=1$, since $\kappa\in X^\star(T)^{M,+}$.

Note that we have now expressed all the small slope conditions as $\lambda\not\geq -w\cdot\nu$ as w ranges over a certain subset of W. We may use this to compare them.

Proposition 5.61. *Let $\kappa\in X^\star(T)^{M,+}$ and let $w\in C(\kappa)^+$. Let $v = w_{0,M}w$ and let ν be given by $v(\nu+\rho) = -\kappa-\rho$ so that $\nu+\rho\in X^\star(T)^+_{\mathbb{R}}$. Then a slope $\lambda\in X^\star(T^d)_{\mathbb{R}}$ satisfies $+, ss(\nu)$ if and only if it satisfies both $+, ss^M(\kappa)$ and $+, ss_{M,w}(\kappa)$.*

Proof. We apply the characterizations of propositions 5.56, 5.57, 5.59. Then it is clear that $+, ss(\nu)$ implies both $+, ss^M(\kappa)$ and $+, ss_{M,w}(\kappa)$. For the other direction, we need to show that for each $\alpha\in\Delta$, then either $s_\alpha\in({}^MW)^{-1}C(\kappa)^+$ or $vs_\alpha v^{-1}\in W_M$.

We apply lemma 5.62 to $w\in {}^MW$ and s_α. If $ws_\alpha\in {}^MW$ then $s_\alpha = (ws_\alpha)^{-1}w\in({}^MW^{-1})w$. Otherwise there exists $\beta\in\Delta_M$ so that $s_\beta w = ws_\alpha$ and hence $vs_\alpha v^{-1} = w_{0,M}^{-1}s_\beta w_{0,M}\in W_M$. □

Lemma 5.62. *Let $w\in {}^MW$ and $\alpha\in\Delta$. Then either $ws_\alpha\in {}^MW$ or $ws_\alpha = s_\beta w$ for $\beta\in\Delta_M$.*

Proof. If $ws_\alpha\notin {}^MW$ then there exists $\beta\in\Delta_M$ with $s_\alpha(w^{-1}(\beta)) = (ws_\alpha)^{-1}(\beta)\in\Phi^-$. But $w^{-1}(\beta)\in\Phi^+$. Hence $w^{-1}(\beta)=\alpha$, so $w^{-1}s_\beta w = s_\alpha$, hence $ws_\alpha = s_\beta w$. □

We can write $\Phi = \coprod_i \Phi_i$ where the Φ_i are simple root systems. Then $\Phi_M = \coprod_i \Phi_{M,i}$ where $\Phi_{M,i} = \Phi_M \cap \Phi_i$. We let $\Phi_b = \coprod_{i, \Phi_i \neq \Phi_{M,i}} \Phi_i$ be the union of the simple factors where M is a proper levi. Let $\Delta_b = \Phi_b \cap \Delta$.

Proposition 5.63. *Suppose that $\nu \in X^\star(T)^+$. Then the following conditions on a slope $\lambda \in X^\star(T^d)^+_{\mathbb{R}}$ are equivalent.*

1. *λ satisfies $+, ss^M(-w_{0,M}w(\nu+\rho)-\rho)$ for all $w \in {}^MW$.*
2. *$\lambda \ngeq -s_\alpha \cdot \nu$ for all $\alpha \in \Delta_b$.*

Proof. We need to show that for $\alpha \in \Delta$, we have $\alpha \in \Delta_b$ if and only if there is $w \in {}^MW$ so that $ws_\alpha w^{-1} \notin W_M$.

The later condition is equivalent to $w\alpha \notin \Phi_M$ for all $w \in W$ (since if we write $w = w_M w^M$ then $w\alpha \in \Phi_M$ implies $w^M\alpha \in \Phi_M$) but $w\alpha$ for $w \in W$ span $\mathbb{Q}\Phi_i$ where Φ_i is the simple factor containing α and so the only way that we will have $w\alpha \in \Phi_M$ for all $w \in W$ is if $\Phi_i = \Phi_{i,M}$. □

We call a slope λ satisfying these equivalent conditions $+, ss_b(\nu)$. We define $-, ss_b(\nu)$ in the obvious analogous way. These conditions will arise in connection with vanishing theorems for Betti cohomology, deduced from vanishing theorems for coherent cohomology via Faltings' dual BGG spectral sequence.

5.11.2. *The strongly small slope conditions.* We now introduce some slightly stronger versions of the small slope conditions of the last section. We need these because we cannot prove the slope bounds of conjecture 5.29, but only the weaker bounds of theorem 5.33.

Definition 5.64. *Let $\lambda \in X^\star(T^d)_{\mathbb{R}}$.*

- *Let $\kappa \in X^\star(T)^{M,+}$.*
  - *We say λ satisfies $+, sss^M(\kappa)$ if for all $w \in {}^MW \setminus C(\kappa)^+$,*
  $$\lambda \ngeq w^{-1}w_{0,M}\kappa.$$
  - *We say λ satisfies $-, sss^M(\kappa)$ if for all $w \in {}^MW \setminus C(\kappa)^-$,*
  $$\lambda \nleq w^{-1}\kappa.$$
- *Let $\kappa \in X^\star(T)$ and let $w \in {}^MW$.*
  - *We say λ satisfies $+, sss_{M,w}(\kappa)$ if for all $w' \in W_M$, $w' \neq 1$,*
  $$\lambda \ngeq w^{-1}w_{0,M}(w'(\kappa+\rho)-\rho).$$
  - *We say λ satisfies $-, sss_{M,w}(\kappa)$ if for all $w' \in W_M$, $w' \neq 1$,*
  $$\lambda \nleq w^{-1}(w'(\kappa+\rho)-\rho).$$

It is immediate from the definitions that λ satisfies $-, sss^M(\kappa)$ if and only if $w_0(\lambda)$ satisfies $+, sss^M(\kappa)$ and similarly λ satisfies $-, sss_{M,w}(\kappa)$ if and only if $w_0(\lambda)$ satisfies $+, sss_{M,w_{0,M}ww_0}(\kappa)$. However the other symmetry related to duality does not hold. Thus we will also consider the dual conditions:

- We say λ satisfies $\pm, sss^M(\kappa)^\vee$ if $-\lambda$ satisfies $\mp, sss^M(-w_{0,M}\kappa - 2\rho_{nc})$.
- We say λ satisfies $\pm, sss_{M,w}(\kappa)^\vee$ if $-\lambda$ satisfies $\mp, sss_{M,w}(-w_{0,M}\kappa - 2\rho_{nc})$.

We introduce combinations of these conditions motivated by propositions 5.63 and 5.61.

- We say that λ satisfies $+, sss_w(\nu)$ if it satisfies $+, sss^M(-w_{0,M}w(\nu+\rho)-\rho)$ and $+, sss_{M,w}(-w_{0,M}w(\nu+\rho)-\rho)$. We say that λ satisfies $-, sss_w(\nu)$ if it satisfies $-, sss^M(-ww_0(\nu+\rho)-\rho)$ and $-, sss_{M,w}(-ww_0(\nu+\rho)-\rho)$.
- We say that λ satisfies $\pm, sss_b(\nu)$ if it satisfies $\pm, sss^M(-w_{0,M}w(\nu+\rho)-\rho)$ for all $w \in {}^M W$.

We also define the dual conditions in the obvious way.

5.12. **Small slopes, classicality and vanishing.**

5.12.1. *Coherent cohomology.* Theorem 5.33 and the definition of the small slope condition implies the following vanishing.

Corollary 5.65. *Let $\kappa \in X^\star(T^c)^{M_\mu,+}$ and let $\chi : T(\mathbb{Z}_p) \to F^\times$ be a finite order character. Then*

$$\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{\pm,sss^M(\kappa)} = \mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{\pm,sss^M(\kappa)} = 0$$

for $w \notin C(\kappa)^\pm$.

This implies that if we take the strongly small slope part of the spectral sequences from finite slope overconvergent cohomology to finite slope classical cohomology of Theorem 5.15, all the terms for $w \notin C(\kappa)^\pm$ vanish. We immediately deduce our first main classicality theorem.

Theorem 5.66. *Let $\kappa \in X^\star(T^c)^{M_\mu,+}$ and assume that $\kappa+\rho$ is regular so that $C(\kappa)^\pm = \{w_\pm\}$. Let $\chi : T(\mathbb{Z}_p) \to F^\times$ be a finite order character. Then the spectral sequences of Theorem 5.15 induces isomorphisms*

$$\mathrm{R}\Gamma_{w_\pm}(K^p,\kappa,\chi)^{\pm,sss^M(\kappa)} \simeq \mathrm{R}\Gamma(K^p,\kappa,\chi)^{\pm,sss^M(\kappa)}$$

$$\mathrm{R}\Gamma_{w_\pm}(K^p,\kappa,\chi,cusp)^{\pm,sss^M(\kappa)} \simeq \mathrm{R}\Gamma(K^p,\kappa,\chi,cusp)^{\pm,sss^M(\kappa)}$$

Remark 5.67. Cases of this theorem for the degree 0 cohomology of PEL Shimura varieties were already proven. See for example [Col96], [Kas06], [Pil11], [BPS16].

The following corollary gives a situation for which the interior Cousin complex computes the classical interior cohomology.

Corollary 5.68. *Let $\kappa \in X^\star(T^c)^{M_\mu,+}$ and assume that $\kappa+\rho$ is regular. Then $\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^{\pm,sss^M(\kappa),sss^M(\kappa)^\vee}$ computes $\overline{\mathrm{H}}^\star(K^p,\kappa,\chi)^{\pm,sss^M(\kappa),sss^M(\kappa)^\vee}$.*

Proof. We have $C(\kappa)^\pm = \{w_\pm\}$. We deduce that

$$\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^{\pm,sss^M(\kappa),sss^M(\kappa)^\vee} =$$

$$\mathrm{Im}\big(\mathrm{H}^{\ell_\pm(w)}_w(K^p,\kappa,\chi,cusp)^{\pm,sss^M(\kappa),sss^M(\kappa)^\vee} \to$$

$$\mathrm{H}^{\ell_\mp(w)}_w(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,sss^M(\kappa),sss^M(\kappa)^\vee,\vee}\big)[-\ell_\pm w]$$

which computes $\overline{\mathrm{H}}^\star(K^p,\kappa,\chi)^{\pm,sss^M(\kappa),sss^M(\kappa)^\vee}$ by theorem 5.66. □

We also deduce vanishing theorems for classical cohomology.

Theorem 5.69. *Let $\kappa \in X^\star(T^c)^{M_\mu,+}$ and let $\chi : T(\mathbb{Z}_p) \to F^\times$ be a finite order character.*

(1) $\mathrm{R}\Gamma(K^p,\kappa,\chi,cusp)^{\pm,sss^M(\kappa)}$ *and* $\mathrm{R}\Gamma(K^p,\kappa,\chi,cusp)^{\pm,sss^M(\kappa)^\vee}$ *are concentrated in degree* $[0,\ell_{\max}(\kappa)]$.

(2) $\mathrm{R}\Gamma(K^p, \kappa, \chi)^{\pm, sss^M(\kappa)}$ *and* $\mathrm{R}\Gamma(K^p, \kappa, \chi)^{\pm, sss^M(\kappa)^\vee}$ *are concentrated in degree* $[\ell_{\min}(\kappa), d]$.

(3) $\overline{\mathrm{H}}^i(K^p, \kappa, \chi, cusp)^{\pm, sss^M(\kappa)}$ *and* $\overline{\mathrm{H}}^i(K^p, \kappa, \chi, cusp)^{\pm, sss^M(\kappa)^\vee}$ *are concentrated in degree* $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$.

Proof. The vanishing result for $\pm, sss^M(\kappa)$ follows from the spectral sequences of theorem 5.15 and the vanishing results of propositions 5.19 and 5.25. The vanishing for the dual condition $\pm, sss^M(\kappa)^\vee$ follows by Serre duality. □

Remark 5.70. If we assume the conjecture 5.29, then the strongly small slope conditions in theorems 5.69 can be weakened to small slope condition. In theorem 6.49 we will actually be able to prove the theorems for the small slope condition for the interior cohomology only, using the eigenvariety.

Remark 5.71. In [Lan16], an analog of theorem 5.69 is proved without any small slope condition, but with a regularity condition on the weights κ and ν instead.

Remark 5.72. The classical coherent cohomology can be computed in terms of automorphic forms for G by the result of Su [Su18]. Thus it may be possible to reprove theorem 5.69 with sufficient knowledge of automorphic forms on G.

5.12.2. *Betti cohomology.* Let $\nu \in X^\star(T^c)^+$. We let W_ν be the corresponding irreducible representation of G with highest weight ν and $W_\nu^\vee$ be its contragrediant. We have an associated local system $\mathcal{W}_\nu^\vee$ over $S_K(\mathbb{C})$.

Using Faltings' dual BGG spectral sequence we deduce vanishing results for the small slope parts of Betti cohomology.

Theorem 5.73. *Let* $\nu \in X^\star(T/Z_s(G))^+$.

(1) $\mathrm{H}^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, sss_b(\nu)}$ *and* $\mathrm{H}^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, sss_b(\nu)^\vee}$ *are concentrated in degree* $[d, 2d]$.

(2) $\mathrm{H}_c^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, sss_b(\nu)}$ *and* $\mathrm{H}_c^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, sss_b(\nu)^\vee}$ *is concentrated in degree* $[0, d]$.

(3) $\overline{\mathrm{H}}^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, sss_b(\nu)}$ *and* $\overline{\mathrm{H}}^i(S_K(\mathbb{C}), \mathcal{W}_\nu^\vee)^{\pm, sss_b(\nu)^\vee}$ *are concentrated in degree* d.

Remark 5.74. If we assume the conjecture 5.29, then the strongly small slope conditions in 5.73 can be weakened to small slope condition. In theorem 6.49 we will actually be able to prove the theorems for the small slope condition for the interior cohomology only, using the eigenvariety.

Remark 5.75. In [Lan16], analogs of theorem 5.73 are proved without any small slope condition, but with a regularity condition on the weight ν instead.

Remark 5.76. The classical Betti cohomology can be computed in terms of automorphic forms for G by the results of Franke [Fra98]. Thus it may be possible to reprove theorem 5.73 with sufficient knowledge of automorphic forms on G.

5.12.3. *Small slope conditions, Jacquet modules.* We now use these small slope condition to define certain direct summands of smooth admissible representations and apply this to the cohomology of the Shimura variety.

Proposition 5.77. *Let* π *be a smooth admissible representation of* $G(\mathbb{Q}_p)$ *and let* $\nu \in X^\star(T)$ *satisfy* $\nu + \rho \in X^\star(T)^+_{\mathbb{R}}$. *Then the following are equivalent:*

(1) *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},+,ss(\nu)} \neq 0$.
(2) *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},-,ss(\nu)} \neq 0$.
(3) $(\pi_U)^{+,ss(\nu)} \neq 0$.
(4) $(\pi_{\overline{U}})^{-,ss(\nu)} \neq 0$.

We have the same equivalent properties when the condition $ss(\nu)$ *replaced by* $ss_b(\nu)$, $sss_b(\nu)$, *or* $ss^M(\kappa)$, $sss^M(\kappa)$ *for* $\kappa \in X^\star(T)^{M,+}$.

Proof. The equivalence of (1) with (3) and (2) with (4) is immediate from proposition 4.20. The equivalence of 3 and 4 follows from proposition 5.54 and the isomorphism between π_U and $\pi_{\overline{U}}$ given by w_0. □

Definition 5.78. *Let* π *be a smooth admissible representation of* $G(\mathbb{Q}_p)$, *let* $\nu \in X^\star(T)$ *satisfy* $\nu+\rho \in X^\star(T)^+_{\mathbb{R}}$ *and let* $\kappa \in X^\star(T)^{M,+}$. *We define* $\pi^{ss(\nu)} \subseteq \pi$ *to be the sum of all indecomposable summands of* π *which satisfy the equivalent conditions of proposition 5.77 for* $ss(\nu)$. *We define similarly* $\pi^{ss_b(\nu)} \subseteq \pi$ *and* $\pi^{ss^M(\kappa)} \subseteq \pi$ *as the sum of all indecomposable summands of* π *which satisfy the equivalent conditions of proposition 5.77 for* $ss_b(\nu)$ *or* $ss^M(\kappa)$.

Remark 5.79. It is not necessarily true that any irreducible factor of $\pi^{ss(\nu)}$ satisfies the condition $ss(\nu)$ and similarly for the condition $ss_b(\nu)$ or $ss^M(\kappa)$.

Remark 5.80. If π is irreducible, then $\pi^{ss(\nu)} = \pi$ means that π admits an embedding in $\iota_B^G\psi$ for a character $\psi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_p^\times$ with $v(\psi)$ satisfying $+, ss(\nu)$. A similar remark holds for the other slope conditions.

We now turn to the strongly small slope condition. Here we take the infimum of the strongly small slope condition for a weight and the dual condition.

Definition 5.81. *Let* π *be a smooth admissible representation of* $G(\mathbb{Q}_p)$ *and let* $\nu \in X^\star(T)$ *satisfy* $\nu+\rho \in X^\star(T)^+_{\mathbb{R}}$ *and let* $\kappa \in X^\star(T)^{M,+}$. *We define* $\pi^{sss_b(\nu)} \subseteq \pi$ *to be the sum of all indecomposable summands of* π *which satisfy either :*

(1) *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},+,sss_b(\nu)} \neq 0$.
(2) *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},+,sss_b(\nu)^\vee} \neq 0$.

We define $\pi^{sss^M(\kappa)} \subseteq \pi$ *to be the sum of all indecomposable summands of* π *which satisfy either :*

(1) *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},+,sss^M(\kappa)} \neq 0$.
(2) *There exists* $m \geq b \geq 0$ *such that* $\pi^{K_{p,m,b},+,sss^M(\kappa)^\vee} \neq 0$.

With these definition in place, we can deduce (most of) theorems 1.5 and 1.10 of the introduction (we will be able to use the small slope condition for interior cohomology after we prove theorem 6.49) :

Theorem 5.82. *For any* $\kappa \in X_\star(T)^{M_\mu,+}$,

(1) $\overline{\mathrm{H}}^i(K^p,\kappa)^{sss^M(\kappa)}$ *is concentrated in the range* $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$,
(2) $\mathrm{H}^i(K^p,\kappa,cusp)^{sss^M(\kappa)}$ *is concentrated in the range* $[0, \ell_{\max}(\kappa)]$,
(3) $\mathrm{H}^i(K^p,\kappa)^{sss^M(\kappa)}$ *is concentrated in the range* $[\ell_{\min}(\kappa), d]$.

For any $\nu \in X_\star(T^c)^+$,

(1) $\overline{\mathrm{H}}^i(K_p,\mathcal{W}_\nu^\vee)^{sss_b(\nu)}$ *is concentrated in the middle degree* d,
(2) $\mathrm{H}_c^i(K^p,\mathcal{W}_\nu^\vee)^{sss_b(\nu)}$ *is concentrated in the range* $[0,d]$,
(3) $\mathrm{H}^i(K^p,\mathcal{W}_\nu^\vee)^{sss_b(\nu)}$ *is concentrated in the range* $[d,2d]$.

Proof. This follows immediately from theorems 5.69 and 5.73. □

5.13. **De Rham and rigid cohomology.** Let $\nu \in X^\star(T^c)^+$. Let $(\mathcal{W}^\vee_{\nu,dR}, \nabla)$ be the associated filtered vector bundle with integrable logarithmic connection over $S^{tor}_{K,\Sigma}$. Over $S_K(\mathbb{C})$, the set of horizontal sections of the corresponding holomorphic vector bundle is the local system $\mathcal{W}^\vee_\nu$. Let $\mathrm{DR}(\mathcal{W}^\vee_\nu) = \mathcal{W}^\vee_{\nu,dR} \otimes_{\mathscr{O}_{S^{tor}_{K,\Sigma}}} \Omega^\bullet_{S^{tor}_{K,\Sigma}/F}(\log(D))$ be the filtered de Rham complex with logarithmic poles associated to $(\mathcal{W}^\vee_{\nu,dR}, \nabla)$. Its cohomology will be denoted $\mathrm{R}\Gamma_{dR}(S^{tor}_{K,\Sigma}, \mathcal{W}^\vee_\nu)$. We also consider the sub-complex $\mathrm{DR}(\mathcal{W}^\vee_\nu(-D))$ and its cohomology will be denoted $\mathrm{R}\Gamma_{dR,c}(S^{tor}_{K,\Sigma}, \mathcal{W}^\vee_\nu)$ where the supscript c stands for compact support.

Faltings' dual BGG complex for $\mathcal{W}^\vee_\nu$ is a filtered complex $\mathrm{BGG}(\mathcal{W}^\vee_\nu)$ in the category of vector bundles with maps given by differential operators. See for example [LP18] or [LLZ19], sect. 6.1. We have $\mathrm{BGG}(\mathcal{W}^\vee_\nu)^i = \oplus_{w \in {}^MW, \ell(w)=i} \mathcal{V}_{-ww_0(\nu+\rho)-\rho}$. We also have a subcomplex $\mathrm{BGG}(\mathcal{W}^\vee_\nu(-D))$ with

$$\mathrm{BGG}(\mathcal{W}^\vee_\nu(-D))^i = \oplus_{w \in {}^MW, \ell(w)=i} \mathcal{V}_{-ww_0(\nu+\rho)-\rho}(-D).$$

We have (see for example [LLZ19], thm. 6.1.10) :

Theorem 5.83. *There is a filtered quasi-isomorphism* $\mathrm{BGG}(\mathcal{W}^\vee_\nu) \to \mathrm{DR}(\mathcal{W}^\vee_\nu)$ *in the category of vector bundles over* $S^{tor}_{K,\Sigma}$*, with morphisms given by differential operators. The stupid filtration on* $\mathrm{BGG}(\mathcal{W}^\vee_\nu)$ *induces is a spectral sequence*

$$E_1^{p,q} = \oplus_{w \in {}^MW, \ell(w)=p} \mathrm{H}^q(S^{tor}_{K,\Sigma}, \mathcal{V}_{-ww_0(\nu+\rho)-\rho}) \Rightarrow \mathrm{H}^{p+q}_{dR}(S^{tor}_{K,\Sigma}, \mathcal{W}^\vee_\nu)$$

degenerating at E_1*. There is a quasi-isomorphism* $\mathrm{BGG}(\mathcal{W}^\vee_\nu(-D)) \to \mathrm{DR}(\mathcal{W}^\vee_\nu(-D))$*. The stupid filtration on* $\mathrm{BGG}(\mathcal{W}^\vee_\nu(-D))$ *induces a spectral sequence*

$$E_1^{p,q} = \oplus_{w \in {}^MW, \ell(w)=p} \mathrm{H}^q(S^{tor}_{K,\Sigma}, \mathcal{V}_{-ww_0(\nu+\rho)-\rho}(-D)) \Rightarrow \mathrm{H}^{p+q}_{dR,c}(S^{tor}_{K,\Sigma}, \mathcal{W}^\vee_\nu)$$

degenerating at E_1*.*

Remark 5.84. Instead of using the stupid filtration on $\mathrm{BGG}(\mathcal{W}^\vee_\nu)$ one can use the filtration F corresponding to the Hodge filtration on $\mathrm{DR}(\mathcal{W}^\vee_\nu)$. The associated graded of this filtration F are complexes of automorphic vector bundles (featuring those appearing as objects in $\mathrm{BGG}(\mathcal{W}^\vee_\nu)$), with trivial differential. The spectral sequence for the F-filtration is then the Hodge-to-de Rham spectral sequence. It also degenrates at E_1. The difference between the Hodge-to-de Rham spectral sequence and the stupid filtration spectral sequence is therefore basically a reindexing of the terms.

We now pass to p-adic geometry. We can consider $\mathrm{DR}(\mathcal{W}^\vee_\nu)$ and $\mathrm{BGG}(\mathcal{W}^\vee_\nu)$ as complex of vector bundles with maps given by differential operators over the adic space $\mathcal{S}^{tor}_{K,\Sigma}$ and the GAGA theorem ensures that $\mathrm{R}\Gamma_{dR}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{W}^\vee_\nu)$ is still computing the algebraic de Rham cohomology groups. If $K = K_pK^p$ if $K_p = K_{p,m,b}$, and $\chi : T_b(\mathbb{Z}_p) \to F^\times$ is a character, we can define $\mathrm{R}\Gamma_{dR}(K^p, \mathcal{W}^\vee_\nu, \chi)^{\pm, fs}$ as a direct factor of the complex of $\mathrm{R}\Gamma_{dR}(\mathcal{S}^{tor}_{K_pK^p,\Sigma}, \mathcal{W}^\vee_\nu)$ (see section 4.2.4). We can also define $\mathrm{R}\Gamma_{dR,c}(K^p, \mathcal{W}^\vee_\nu, \chi)^{\pm, fs}$ as a direct factor of the complex of $\mathrm{R}\Gamma_{dR,c}(\mathcal{S}^{tor}_{K_pK^p,\Sigma}, \mathcal{W}^\vee_\nu)$

For any $w \in {}^MW$, we can also make sense of $\mathrm{R}\Gamma_{dR,w}(K^p, \mathcal{W}^\vee_\nu, \chi)^{\pm, fs}$ as in section 5.4. Namely, one just copies verbatim this section with the automorphic sheaf $\mathcal{V}_\kappa$ replaced by the complex of automorphic sheaves $\mathrm{DR}(\mathcal{W}^\vee_\nu)$. Similarly, we can define $\mathrm{R}\Gamma_{dR,c,w}(K^p, \mathcal{W}^\vee_\nu, \chi)^{\pm, fs}$ by considering the cohomology of the complex $\mathrm{DR}(\mathcal{W}^\vee_\nu(-D))$.

Remark 5.85. It would be interesting to study in depth the cohomologies
$$\mathrm{R}\Gamma_{dR,w}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}.$$
For example, are the cohomology groups finite dimensional F-vector spaces? In the Siegel case, the cohomologies $\mathrm{R}\Gamma_{dR,w}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}$ and $\mathrm{R}\Gamma_{dR,c,w}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}$ for $w \in \{Id, w_0^M\}$ are rigid cohomologies with support conditions of certain coverings of the ordinary locus in the Shimura variety. Is this a general phenomena (with the ordinary locus replaced by the Igusa variety corresponding to w)?

The following theorem is the direct analogue of theorem 5.15. The proof proceeds in the exact same way.

Theorem 5.86. *Let $\nu \in X^\star(T^c)^+$ be a weight and let $\chi : T(\mathbb{Z}_p) \to F^\times$ be a finite order character. For a choice of $+$ or $-$, there is a $\mathcal{H}^{\pm}_{p,m,b}$-equivariant spectral sequence $\mathbf{E}^{p,q}_{dR}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm}$ converging to classical finite slope de Rham cohomology $\mathrm{H}^{p+q}_{dR}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}$, such that*
$$\mathbf{E}^{p,q}_{dR,1}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm} = \oplus_{w \in {}^MW, \ell_{\pm}(w) = p} \mathrm{H}^{p+q}_{dR,w}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}.$$
There are also spectral sequences $\mathbf{E}^{p,q}_{dR,c}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm}$ converging to $\mathrm{H}^{p+q}_{dR,c}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}$ such that
$$\mathbf{E}^{p,q}_{dR,c,1}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm} = \oplus_{w \in {}^MW, \ell_{\pm}(w) = p} \mathrm{H}^{p+q}_{dR,c,w}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}.$$

It follows that we have two spectral sequences converging to the classical cohomology $\mathrm{R}\Gamma_{dR}(K^p, \mathcal{W}_\nu^\vee, \chi)^{\pm, fs}$ (and the compactly supported one). The first one is associated to the stupid filtration on the de Rham complex (and is basically the Hodge-to-de Rham spectral sequence). The other one is the spectral sequence of theorem 5.86 and it is really coming from the Bruhat stratification on the Flag variety.

By comparing both spectral sequences on the stronly small slope part, we obtain the following decomposition of the de Rham cohomology.

Theorem 5.87. *For all $\nu \in X^\star(T^c)^+$, we have that :*
$$(\mathrm{H}^n_{dR}(K^p, \mathcal{W}_\nu^\vee, \chi)^{+, sss_b(\nu)}) = \bigoplus_{p+q=n} \oplus_{w, \ell(w) = p} \mathrm{H}^q(K^p, -ww_0(\nu + \rho) - \rho, \chi)^{+, sss_b(\nu)}$$
and that
$$(\mathrm{H}^n_{dR}(K^p, \mathcal{W}_\nu^\vee, \chi))^{-, sss_b(\nu)} = \bigoplus_{p+q=n} \oplus_{w, \ell_-(w) = p} \mathrm{H}^q(K^p, -ww_0(\nu + \rho) - \rho, \chi)^{-, sss_b(\nu)}$$
We have similarly:
$$(\mathrm{H}^n_{dR,c}(K^p, \mathcal{W}_\nu^\vee, \chi)^{+, sss_b(\nu)}) = \bigoplus_{p+q=n} \oplus_{w, \ell(w) = p} \mathrm{H}^q(K^p, -ww_0(\nu+\rho)-\rho, \chi, cusp)^{+, sss_b(\nu)}$$
and that
$$(\mathrm{H}^n_{dR,c}(K^p, \mathcal{W}_\nu^\vee, \chi))^{-, sss_b(\nu)} = \bigoplus_{p+q=n} \oplus_{w, \ell_-(w) = p} \mathrm{H}^q(K^p, -ww_0(\nu+\rho)-\rho, \chi, cusp)^{-, sss_b(\nu)}.$$

Proof. We only prove the first displayed equation. The idea of the proof is that the two spectral spectral sequences are in a certain sense opposite to each other on the strongly small slope part, and therefore, not only do we have degeneration at E_1, but also the induced filtration on cohomology is split.

From theorem 5.66 and corollary 5.65, we have that

$$\begin{aligned}\mathrm{H}^{p+q}_{dR,w}(K^p,\mathcal{W}^\vee_\nu,\chi)^{+,sss_b(\nu)} &= \mathrm{H}^{p+q-\ell_-(w)}_w(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,sss_b(\nu)}\\ &= \mathrm{H}^{p+q-\ell_-(w)}(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,sss_b(\nu)}\end{aligned}$$

For the spectral sequence of theorem 5.86, we deduce that

$$\begin{aligned}\mathbf{E}^{p,q}_{dR,1}(K^p,\mathcal{W}^\vee_\nu,\chi)^{+,sss_b(\nu)} &= \oplus_{w\in {}^MW,\ell(w)=p}\mathrm{H}^{p+q}_{dR,w}(K^p,\mathcal{W}^\vee_\nu,\chi)^{\pm,fs}\\ &= \oplus_{w\in {}^MW,\ell(w)=p}\mathrm{H}^{p+q-\ell_-(w)}(K^p,-w_{0,M}w(\nu+\rho)-\rho,\chi)^{+,sss_b(\nu)}\end{aligned}$$

For the spectral sequence of theorem 5.83, we have

$$\begin{aligned}(E^{p,q}_1)^{+,sss_b(\nu)} &= \oplus_{w\in {}^MW,\ell(w)=p}\mathrm{H}^q(K^p,-ww_0(\nu+\rho)-\rho,\chi)^{+,sss_b(\nu)}\\ &= \mathbf{E}^{p',q'}_{dR,1}(K^p,\mathcal{W}^\vee_\nu,\chi)^{+,sss_b(\nu)},\ \text{for } p'=d-p,\ p'+q'=p+q.\end{aligned}$$

The spectral sequence of theorem 5.83 degenerates at E_1, and we deduce that we get two opposite filtrations on the cohomology. This gives the splitting. □

Remark 5.88. This theorem is reminiscent of complex Hodge theory, where one obtains a splitting of the Hodge filtration given by harmonic $\mathcal{C}^\infty$-differential forms. In the p-adic setting, the de Rham cohomology comes with additional structure (notably a Frobenius on the Hyodo-Kato cohomology [HK94]). Is the above splitting induced by these additional structure?

We conclude this section by constructing an interior Cousin bi-complex analogue to the interior Cousin complex of section 5.8.

We define $\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm$ as the bi-complex concentrated in degrees in $[0,d]\times[0,d]$, where for all $(i,j)\in\mathbb{Z}\times\mathbb{Z}$, we have :

$$(\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm)^{(i,j)} = \oplus_{w\in {}^MW,\ell_\pm(w)=i}\oplus_{w'\in {}^MW,\ell(w')=j}\overline{\mathrm{H}}^i_w(K^p,-w'w_0(\nu+\rho)-\rho,\chi)^{\pm,fs}$$

The horizontal complexes $(\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm)^{(\bullet,j)}$ are

$$\oplus_{w'\in {}^MW,\ell(w')=j}\overline{\mathcal{C}ous}(K^p,-w'w_0(\nu+\rho)-\rho,\chi)^\pm$$

and the vertical differentials are those given by the maps in the BGG complexes.

Let us define the interior de Rham cohomology by

$$\overline{\mathrm{H}}^i_{dR}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm = \mathrm{Im}(\mathrm{H}^i_{dR,c}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm \to \mathrm{H}^i_{dR,c}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm).$$

We find:

Proposition 5.89. *The cohomology* $\overline{\mathrm{H}}^i_{dR}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm$ *is a subquotient of*

$$\mathrm{H}^i(\mathrm{Tot}(\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm)).$$

Proof. This follows from corollary 5.27. □

Remark 5.90. When the Shimura variety is compact, conjecture 5.20 implies that $\mathrm{Tot}(\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm)$ is quasi-isomorphic to $\mathrm{R}\Gamma_{dR}(K^p,\mathcal{W}^\vee_\nu,\chi)^\pm$.

5.14. **Explicit formulas in the symplectic case.** Let V be a $\mathbb{Q}$-vector space of dimension $2g$. Let Ψ be the symplectic form on V given in the canonical basis $e_1, \cdots, e_{2g}$ by $\Psi(e_i, e_j) = 1$ if $i \leq g$ and $j = 2g-i+1$ and $\Psi(e_i, e_j) = 0$ if $i \leq g$ and $j \neq 2g-i+1$. Let $G = \mathrm{GSp}_{2g}$, be the subgroup of automorphisms of V respecting Ψ up to a similitude factor ν. We pick the maximal diagonal torus T in G. A typical element $t \in T$ is labelled $(t_1, \cdots, t_g; c) = \mathrm{diag}(t_1 c, \cdots, t_g c, t_g^{-1} c, \cdots, t_1^{-1} c)$. The character group $X^\star(T)$ identifies with $\{(k_1, \cdots, k_g; k) \in \mathbb{Z}^{g+1}, \sum k_i = k \mod 2\}$. The action is given by $(k_1, \cdots, k_g; k)(t_1, \cdots, t_g; c) = \prod t_i^{k_i} c^k$. We let P_μ^{std} be the stabilizor of the Lagrangian plan $\langle e_1, \cdots, e_g\rangle$. We therefore let P_μ be the stabilizor of the Lagrangian plan $\langle e_{g+1}, \cdots, e_{2g}\rangle$. We have $M_\mu \simeq \mathrm{GL}_g \times \mathbb{G}_m$. We choose the upper triangular Borel in M_μ, which fixes the positive compact roots. Recall that we choose the positive non-compact roots to be in $\mathfrak{g}/\mathfrak{p}_\mu^{std}$. It follows that $X^\star(T)^{M_\mu,+} = \{(k_1, \cdots, k_g; k), k_1 \geq k_2 \geq \cdots \geq k_g\}$ and $X^\star(T)^+ = \{(k_1, \cdots, k_g; k), 0 \geq k_1 \geq k_2 \geq \cdots \geq k_g\}$. We have $\rho = (-1, -2, -3, \cdots, g; 0)$. The usual way to normalize the central character in the theory of Siegel moduler forms is to consider weights of the form $(k_1, \cdots, k_g; -\sum k_i)$, with $k_1 \geq \cdots \geq k_g$. For example (consult [FP19], example 5.2 for the details), when $g = 1$, $\mathcal{V}_{(k;-k)} = \omega_E^{\otimes k}$ where $E \to S_K^\star$ is the universal semi-abelian scheme and ω_E is its co-normal sheaf. When $g = 2$, $\mathcal{V}_{(k_1,k_2;-k_1-k_2)} = \mathrm{Sym}^{k_1-k_2}\omega_A \otimes \det^{k_2}\omega_A$ where $A \to S_{K,\Sigma}^{tor}$ is the universal semi-abelian scheme and ω_A is its co-normal sheaf.

A standard basis of Hecke operators in $\mathcal{H}_{p,1,0}^+$ is given by the classes:

(1) $U_g = [K_{p,1,0}(-1/2, \cdots, -1/2; -1/2)(p)K_{p,1,0}]$, where $(-1/2, \cdots, -1/2; -1/2)(p) = \mathrm{diag}(p^{-1}, \cdots, p^{-1}, 1, \cdots, 1)$.
(2) $U_i = [K_{p,1,0}(0, \cdots, 0, -1, \cdots, -1; -1)(p)K_{p,1,0}]$ for $1 \leq i \leq g-1$ with i many -1 before the ;. We have that $(0, \cdots, 0, -1, \cdots, -1; -1)(p) = \mathrm{diag}(p^{-1}\mathrm{Id}_{g-i}, p^{-2}\mathrm{Id}_i, \mathrm{Id}_i, p\mathrm{Id}_{g-i})$
(3) $S = [pK_{p,1,0}]$, S^{-1}.

Remark 5.91. Following [FP19], remark 5.6, we justify that for $g = 1$, the double class $[K_{p,1,0}\mathrm{diag}(p^{-1}, 1)K_{p,1,0}]$ indeed corresponds to the standard U_p-operator ! For simplicity let us assume that $K^p \subseteq \mathrm{GL}_2(\prod_{\ell \neq p} \mathbb{Z}_\ell)$. The corresponding moduli space (ignoring cusps) parametrizes two elliptic curves E_1, E_2 up to isomorphisms, with K level structures, together with a quasi isogeny $E_2 \to E_1$ giving a map $V(E_2) \to V(E_1)$ where $V(E_i) = \lim E_i[N] \otimes \mathbb{Q}$ is the adelic Tate module, and we ask that this map is represented by $K\mathrm{diag}(p^{-1}, 1)K$. Concretely this means that the quasi-isogeny $E_2 \to E_1$ comes from a degree p isogeny $E_1 \to E_2$ and that this isogeny matches K-level structures on the Tate modules. The $K_{p,1,0}$ level structure is the data of a rank p-subgroup $H_i \subseteq E_i[p]$. Because we choose the lower triangular Borel, we find that the isogeny $E_1 \to E_2$ induces and isomorphism between H_1 and H_2.

5.14.1. *$GL_2/\mathbb{Q}$.* In this case, everything is already in [BP20]. Let $\kappa = (k; -k)$. The Cousin complex is $\mathcal{C}ous(K^p, \kappa, \chi) : \mathrm{H}_1^0(K^p, \kappa, \chi)^{+,fs} \to \mathrm{H}_w^1(K^p, \kappa, \chi)^{+,fs}$ where $\mathrm{H}_1(K^p, \kappa, \chi)^{+,fs}$ is the space of finite slope overconvergent modular forms of weight k, nebentypus χ and $\mathrm{H}_w^1(K^p, \kappa, \chi)^{+,fs}$ is the finite slope part of the cohomology with compact support of the dagger space "ordinary locus" in weight k and nebentypus χ. We have $w_{0,M} = 1$ for GL_2. For $w = 1$, we find that $w^{-1}w_{0,M}(\kappa+\rho)+\rho = (k-2; -k)$ and that $\langle(-1/2; -1/2), (k-2; -k)\rangle = 1$. The un-normalized U_p-operator acts on q-expansion by $\sum a_n q^n = p \sum a_{np} q^n$ and has indeed slope greater or equal to 1 on $\mathrm{H}^0_1(K^p,\kappa,\chi)^{+,fs}$. For $w \neq 1$, we find that $w^{-1}w_{0,M}(\kappa+\rho)+\rho = (-k;-k)$ and we get that $-\langle(-1/2;-1/2),(-k;-k)\rangle = k$. On $\mathrm{H}^1_w(K^p,\kappa,\chi)^{+,fs}$, U_p acts like Frobenius, and this explains why it is of slope greater or equal than k. See lemma 5.3 in [BP20]. We deduce from this lemma the classicality theorem.

5.14.2. $GSp_4/\mathbb{Q}$. The Weyl group is generated by the following transposition: $s_0(k_1,k_2;k) = (k_2,k_1;k)$ and $s_1(k_1,k_2;k) = (-k_1,k_2;k)$. The elements of ${}^MW$ are $Id, s_1, s_1s_0, s_1s_0s_1$. We consider the weight $\kappa = (k_1,k_2;-k_1-k_2)$ so that $-w_{0,M}\kappa = (-k_2,-k_1;k_1+k_2)$. The following table indicates the value of the pairing

$$\langle t, w^{-1}w_{0,M}(\kappa+\rho)+\rho\rangle = \langle t, w^{-1}w_{0,M}(\kappa)\rangle + \langle t, w^{-1}w_{0,M}(\rho)+\rho\rangle$$

where $t = (-1/2,-1/2;-1/2)(p)$ or $(0,-1;-1)(p)$, and $w \in {}^MW$.

	Id	s_1	s_1s_0	$s_1s_0s_1$
(-1/2,-1/2;-1/2)	3	k_2+1	k_2+1	k_1+k_2
(0,-1;-1)	k_2+3	k_2+3	$2k_2+k_1$	$2k_2+k_1$

The table giving the pairing $\langle t, w^{-1}w_{0,M}(\kappa)\rangle$ is deduced from this one by replacing the constants by 0:

	Id	s_1	s_1s_0	$s_1s_0s_1$
(-1/2,-1/2;-1/2)	0	k_2	k_2	k_1+k_2
(0,-1;-1)	k_2	k_2	$2k_2+k_1$	$2k_2+k_1$

The difference between these two tables illustrates the difference between conjecture 5.29 (first table) and theorem 5.33 (second table). Let us explain how to interpret this information.

Let us take a weight in the interior of the holomorphic chamber: $k_2 > 2$. We assume that U_2 has slopes $< k_2+1$. This is the "small slope" condition because there are conjecturally no overconvergent cohomology classes for $w \neq 1$ which satisfy this slope condition. Therefore the U_2-slope $< k_2+1$ part of classical cohomology indentifies conjecturally with the overconvergent cohomology for $w=1$. Under the bounds of theorem 5.33, we have to use the strongly small slope condition $< k_2$ instead.

Let us take a weight in the interior of the "H^1 chamber": $k_2 < 2$ and $k_1+k_2 > 3$. The small slope condition is now U_2-slope < 3 and U_1-slope $< k_1+2k_2$. The first condition kills conjecturally the overconvergent cohomology for 1. The second condition kills the overconvergent cohomology for s_0s_1 and $s_0s_1s_0$. Therefore, the small slope classical cohomology identifies conjecturally with the small slope overconvergent cohomology for $w = s_0$. Under the bounds of theorem 5.33, we have to use the strongly small slope condition U_2-slope < 0 instead of < 3. As a final remark, in this case the cuspidal overconvergent cohomology for $w=1$ is in degree 0, and by a q-expansion argument we can indeed show that the U_2-slope is ≥ 3. Therefore we can show that the small slope classical cuspidal cohomology identifies with the small slope cuspidal overconvergent cohomology for $w = s_0$.

6. p-adic families of overconvergent cohomology

6.1. **On the locally analytic BGG resolution.** In this section we recall some basic facts about analytic inductions and BGG resolutions for p-adic groups. Standard references for this material are [Urb11], section 3 and [Jon11]. The notations for this section are as follows. We let F be a finite extension of $\mathbb{Q}_p$ and we let

$M \to \mathrm{Spec}\, \mathcal{O}_F$ be a split reductive group. We fix a maximal torus T and a Borel B containing T. We let $\Phi_M = \Phi_M^+ \coprod \Phi_M^-$ be the root system. We also let $\Delta_M \subset \Phi_M^+$ be the positive simple roots. We denote by W_M the Weyl group, and denote by $\ell : W_M \to \mathbb{Z}_{\geq 0}$ the length function. For all $i \in \mathbb{Z}_{\geq 0}$, we let $W_M^{(i)}$ be the set of elements in W_M of length i. We denote by $w_{0,M}$ the longest element of W_M. We let ρ_M be half the sum of the positive roots. The Weyl group acts on T, $X_\star(T)$ and $X^\star(T)$. We also have the dotted action on $X^\star(T)$ given by $w \cdot \kappa = w(\kappa + \rho_M) - \rho_M$. We let $X_\star(T)^{M,+}$ and $X^\star(T)^{M,+}$ be the cones of dominant cocharacters and characters and we let $X_\star(T)^{M,++}$ and $X^\star(T)^{M,++}$ be the cones of dominant regular cocharacters and characters. We use a $-$ sign to denote the opposite cones. We assume that $T_F = T \times_{\mathrm{Spec}\, \mathcal{O}_F} \mathrm{Spec}\, F$ is in fact defined over $\mathbb{Q}_p$. Namely, there is a torus $T_{\mathbb{Q}_p}$ and an isomorphism $T_{\mathbb{Q}_p} \times \mathrm{Spec}\, F = T_F$. We often drop the subscripts F or $\mathbb{Q}_p$ when the context is clear.

Remark 6.1. In our applications, M will be the Levi M_μ of the group G which is part of the Shimura datum. A slight warning is that the torus $T_{\mathbb{Q}_p}$ will in general be the conjugate of a maximal torus of $G_{\mathbb{Q}_p}$ by an element of the Weyl group of G which is not necessarily rational.

We let $T^d \subseteq T_{\mathbb{Q}_p}$ be the maximal split subtorus. We let $T(\mathbb{Z}_p) \subseteq T(\mathbb{Q}_p)$ be the maximal compact subgroup. There is a valuation map $v : T(\mathbb{Q}_p) \to X_\star(T^d) \otimes \mathbb{Q}$ whose image is a lattice and whose kernel is $T(\mathbb{Z}_p)$. We let $T^{M,+}$ be the inverse image of $X_\star(T^d)^{M,+}$ via v. We define similarly $T^{M,++}$, $T^{M,-}$, $T^{M,--}$.

For any $\kappa \in X^\star(T)$,we let $F(\kappa)$ be the one dimensional F vector space endowed with the action of $T(\mathbb{Q}_p)$ via the character κ. If V is a F-vector space endowed with an action of (a submonoid of) $T(\mathbb{Q}_p)$, we let $V(\kappa) = V \otimes F(\kappa)$.

We let $\mathcal{M}$ be the quasi-compact adic space over $\mathrm{Spa}(F, \mathcal{O}_F)$ attached to M and we denote by $\mathcal{M}_n$ the subgroup of $\mathcal{M}$ of elements reducing to 1 modulo p^n. We define in a similar fashion $\mathcal{T}$, the quasi-compact torus over $\mathrm{Spa}(F, \mathcal{O}_F)$ attached to T and $\mathcal{T}_n$ the subgroup of $\mathcal{T}$ of elements reducing to 1 modulo p^n.

We let $M_1 \subseteq M(\mathcal{O}_F)$ be a closed subgroup possessing an Iwahori decomposition, in the sense that the product map

$$\overline{N}_1 \times T_1 \times N_1 \to M_1$$

is an isomorphism, where $N_1 = M_1 \cap U$, $T_1 = M_1 \cap T$, $\overline{N}_1 = M_1 \cap \overline{U}$ (for U and $\overline{U}$ the unipotent radical of B and the opposite Borel $\overline{B}$ respectively). We also assume that $T_1 = T(\mathbb{Z}_p)$, and that $T^{M,-}$ normalizes $\overline{N}_1$.

6.1.1. *Algebraic inductions.* Let $\kappa \in X^\star(T)^{M,+}$. We have the algebraic representation V_κ of M with highest weight κ. It can be realized as an algebraic induction:

$$\begin{aligned} V_\kappa &= \mathrm{Ind}_B^M(w_{0,M}\kappa) \\ &= \{f : M \to \mathbb{A}^1 \mid f(mb) = (w_{0,M}\kappa)(b^{-1})f(m),\ \forall\ (m,b) \in M \times B\} \end{aligned}$$

We have a left action of M given by $hf(m) = f(h^{-1}m)$.

6.1.2. *Analytic weights.* Let (A, A^+) be a complete Tate algebra over $(F, \mathcal{O}_F)$ and let κ_A be a continuous morphism $T(\mathbb{Z}_p) \to A^\times$. Let $n \in \mathbb{Z}_{\geq 0}$. We say that κ_A is n-analytic if the map κ_A can be extended to a pairing:

$$T(\mathbb{Z}_p)\mathcal{T}_n \times \mathrm{Spa}(A, A^+) \to \mathbb{G}_m^{an}$$

where $T(\mathbb{Z}_p)\mathcal{T}_n$ is the subgroup of $\mathcal{T}$ generated by $T(\mathbb{Z}_p)$ and $\mathcal{T}_n$ (this is a finite union of translates of $\mathcal{T}_n$). We recall that any continuous character κ_A is n-analytic for some n (see [Urb11], lemma 3.2.5. for example).

6.1.3. *Analytic inductions.* Let (A, A^+) be a complete Tate algebra over $(F, \mathcal{O}_F)$ and let $S = \mathrm{Spa}(A, A^+)$. We assume that A is uniform (i.e A^0 is bounded) and equip A with the supremum norm. Let $n_0 \in \mathbb{Z}_{\geq 0}$ be an integer. We now fix a character $\kappa_A : w_{0,M}^{-1} T(\mathbb{Z}_p) w_{0,M} \to A^\times$ which is n_0-analytic, or equivalently a character $w_{0,M}\kappa_A : T(\mathbb{Z}_p) \to A^\times$ which is n_0-analytic.

Remark 6.2. The notation $w_{0,M}\kappa_A$ may seem strange so let us explain it to orient the reader. Let $\kappa \in X^\star(T)$ be an algebraic character of T. Then for any $w \in W_M$, we have $\langle w\kappa, t\rangle = \langle \kappa, w^{-1}t\rangle$ so that $w\kappa(t) = \kappa(w^{-1}tw)$.

For all $n \geq n_0$ we can define

$$V_{\kappa_A}^{n-an} = \mathrm{an} - \mathrm{Ind}_{\mathcal{B}\cap(M_1\mathcal{M}_n)}^{M_1\mathcal{M}_n}(w_{0,M}\kappa_A) =$$

$$\{f : (M_1\mathcal{M}_n)_S \to \mathbb{A}_S^{1,an} \mid f(mb) = (w_{0,M}\kappa_A)(b^{-1})f(m),\ \forall\,(m,b) \in (M_1\mathcal{M}_n)_S \times (\mathcal{B}\cap(M_1\mathcal{M}_n))_S\}.$$

This is a Banach A-module for the supremum norm. We let $V_{\kappa_A}^{n-an,+}$ be the module of elements with supremum norm less or equal than one. The space $V_{\kappa_A}^{n-an}$ carries the following actions of the group $(M_1\mathcal{M}_n)_S$ and of the monoid $T^{M,+}$:

- $hf(m) = f(h^{-1}m)$ for $h, m \in (M_1\mathcal{M}_n)_S$,
- $tf(m) = f(t^{-1}\overline{n}_m t_m t)$ for $t \in T^{M,+}$, $m \in (M_1\mathcal{M}_n)_S$, and $m = \overline{n}_m t_m n_m$ the Iwahori decomposition of m.

These actions respect the submodule $V_{\kappa_A}^{n-an,+}$.

We now include the following lemma for later use. Let M_1^p be the quotient of M_1 by its maximal normal pro p-subgroup.

Lemma 6.3. *The representation $V_{\kappa_A}^{n-an,+} \otimes_{A^+} A^+/A^{++}$ of $M_1\mathcal{M}_n$ is a countable inductive limit of finite projective A^+/A^{++}-submodules V_i stable under the action of $M_1\mathcal{M}_n$, and with the property that the action on V_{i+1}/V_i factors through an action of M_1^p.*

Proof. The character κ_A takes only finitely many values on A^+/A^{++}, therefore we may assume it is constant. We are then reduced to the case that A is a finite field extension of F and A^+/A^{++} is a finite field. The stabilizer of any vector $v \in V_{\kappa_A}^{n-an,+} \otimes_{A^+} A^+/A^{++}$ is open in $M_1\mathcal{M}_n$ and we deduce that $V_{\kappa_A}^{n-an,+} \otimes_{A^+} A^+/A^{++}$ is a countable inductive limit of finite dimensional representations. Since groups of order p have non-zero fixed vectors on finite dimensional representations in characteristic p, the claim follows. □

We also let $V_{\kappa_A}^{lan} = \mathrm{colim}_n V_{\kappa_A}^{n-an}$ be the locally analytic induction.

Lemma 6.4. *The operators $t \in T^{M,++}$ are compact on $V_{\kappa_A}^{n-an}$ and the maps $V_{\kappa_A}^{n-an} \to V_{\kappa_A}^{n+1-an}$ induce isomorphisms on the finite slope part. The slopes of $t \in T^{M,+}$ on $V_{\kappa_A}^{lan,fs}$ are greater or equal than 0.*

Proof. If $t \in T^{M,++}$, one sees easily that the map $t : V_{\kappa_A}^{n-an} \to V_{\kappa_A}^{n-an}$ improves analyticity. In particular, if $\min_{\alpha\in\Delta_M} v(\alpha(t)) \geq 1$, one has a factorization

$$V_{\kappa_A}^{n-an} \to V_{\kappa_A}^{n+1-an} \to V_{\kappa_A}^{n-an}$$

where the first inclusion is compact. Moreover, we deduce from the definition of the action that if $t \in T^{M,+}$, then t preserves the open and bounded submodule $V_{\kappa_A}^{n-an,+}$. $\square$

The space $V_{\kappa_A}^{n-an}$ embeds in $\mathrm{H}^0(M_1\mathcal{M}_n, \mathscr{O}_{M_1\mathcal{M}_n})$ and similarly $V_{\kappa_A}^{lan}$ embeds in

$$\mathrm{colim}_n \mathrm{H}^0(M_1\mathcal{M}_n, \mathscr{O}_{M_1\mathcal{M}_n}).$$

We have a left action of $M_1\mathcal{M}_n$ on $\mathrm{H}^0(M_1\mathcal{M}_n, \mathscr{O}_{M_1\mathcal{M}_n})$ given by $h*f(m) = f(mh)$. Passing to the limit over n and differentiating, we get an action of the Lie algebra $\mathfrak{m}$ of M on $\mathrm{colim}_n \mathrm{H}^0(M_1\mathcal{M}_n, \mathscr{O}_{M_1\mathcal{M}_n})$ which can be extended to an action of the enveloping algebra $\mathcal{U}(\mathfrak{m})$.

Remark 6.5. We are not requiring that $M_1 \subseteq M(\mathcal{O}_F)$ is an open subgroup and M_1 will not always be Zariski dense in M. In our definition of locally analytic induction, we consider analytic functions on neighborhoods on M_1 in $\mathcal{M}$. These functions are not necessarily determined by their restriction to M_1.

6.1.4. *Twist by a finite order character.* We fix a character $\kappa_A : w_{0,M}^{-1}T(\mathbb{Z}_p)w_{0,M} \to A^\times$ which is n-analytic. Let $\omega_{0,M}\chi : M_1 \to F^\times$ be a finite order character. We still denote by $w_{0,M}\chi$ its restriction to $T(\mathbb{Z}_p)$ and by χ the corresponding character of $w_{0,M}^{-1}T(\mathbb{Z}_p)w_{0,M}$ which we assume also to be n-analytic. We also denote by $w_{0,M}\chi$ the corresponding 1-dimensional representation of M_1. We endow it with the trivial action of $T^{M,+}$ We have the following lemma :

Lemma 6.6. *There is a canonical map $V_{\kappa_A}^{n-an} \otimes_F (w_{0,M}\chi)^{-1} \to V_{\kappa_A\otimes\chi}^{n-an}$ which is an isomorphism of $(M_1, T^{M,+})$-modules.*

Proof. The map is defined by sending a tensor $a \otimes b$ to the product ab as functions on $M_1\mathcal{M}_n$. The rest of the lemma follows easily and is left to the reader. $\square$

6.1.5. *Locally algebraic induction.* Let $\kappa \in X^\star(T)^{M,+}$. From κ we obtain a character $\kappa : w_{0,M}^{-1}T(\mathbb{Z}_p)w_{0,M} \to F^\times$ as the composition of the inclusion $w_{0,M}^{-1}T(\mathbb{Z}_p)w_{0,M} \subset T(F)$ and $\kappa : T(F) \to F^\times$.

From the definitions, restriction induces a natural inclusion $\iota : V_\kappa \hookrightarrow V_\kappa^{lan}$. There is an action of M on V_κ, and therefore actions of M_1 and $T^{M,+}$. The map ι is M_1-equivariant, but not $T^{M,+}$-equivariant. More precisely, we have the following formula:

$$\iota(tv) = (w_{0,M}\kappa)(t)t\iota(v)$$

from which we deduce that the map $\iota : V_\kappa \to V_\kappa^{lan}(w_{0,M}\kappa)$ is $(M_1, T^{M,+})$-equivariant.

We can define the subspace of V_κ^{lalg} of V_κ^{lan}, consisting of elements arising from functions on $M_1\mathcal{M}_n$ for some n which on each component of $M_1\mathcal{M}_n$ are the restriction of a polynomial function on M. This is a $(M_1, T^{M,+})$-subrepresentation. This space contains the space $V_\kappa(-w_{0,M}\kappa)$ of algebraic functions. Let

$$V_{\mathbf{1}}^{sm} = \mathrm{sm} - \mathrm{Ind}_{B\cap M_1}^{M_1}\mathbf{1} =$$

$$\{f : M_1 \to \mathcal{O}_F \mid f(mb) = f(m) \ \forall \ (m,b) \in M_1 \times B \cap M_1, \ f \text{ is locally constant}\}$$

The map $V_\kappa \otimes V_{\mathbf{1}}^{sm} \to V_\kappa^{lalg}(w_{0,M}\kappa)$ is an isomorphism of (M_1, T_1^+)-representations.

6.1.6. *The BGG complex.* For all $\alpha \in \Delta_M$, we fix a generator X_α of the root space $\mathfrak{u}_\alpha \subseteq \mathfrak{m}$ and we have the corresponding generator $X_{-\alpha}$ of $\mathfrak{u}_{-\alpha}$. Let $\kappa \in X^\star(T)$.

Lemma 6.7. *For all $\alpha \in \Delta_M$ such that $\langle \kappa, \alpha^\vee \rangle \geq -1$, we have maps:*

$$\begin{aligned} \Theta_\alpha : V_\kappa^{lan}(w_{0,M}\kappa) &\rightarrow V_{s_\alpha \cdot \kappa}^{lan}(w_{0,M}(s_\alpha \cdot \kappa)) \\ f &\mapsto X_{w_{0,M}\alpha}^{\langle \kappa, \check{\alpha} \rangle + 1} * f \end{aligned}$$

equivariant for the action of $(M_1, T^{M,+})$.

Proof. These maps are constructed in [Urb11], proposition 3.2.11, remark 3.3.11, proposition 3.3.12, as well as [Jon11], section 5, see also the remark below theorem 13. □

Remark 6.8. The normalization of [Urb11] and [Jon11] is slightly different from the one we use here. They realize the highest weight κ representation V_κ as the following induction: $V'_\kappa = \{f : M \rightarrow \mathbb{A}^1 \mid f(\bar{b}m) = \kappa(\bar{b})f(m),\ \forall\ (m, \bar{b}) \in M \times B\}$. To translate to our setting, one simply applies the involution $m \mapsto w_{0,M}m^{-1}w_{0,M}$. We therefore get an isomorphism $V_\kappa \rightarrow V'_\kappa$ which sends f to f' defined by $f'(m) = f(w_{0,M}m^{-1}w_{0,M})$. There is an action of M on V'_κ induced from the right translation action of M on itself. We find that $mf' = ((w_{0,M}mw_{0,M}^{-1})f)'$. This explains the twist by $w_{0,M}$ appearing in lemma 6.7 compared to *loc. cit.*

Remark 6.9. We have

$$w_{0,M}(s_\alpha \cdot \kappa - \kappa) = -(\langle \kappa, \alpha^\vee \rangle + 1)w_{0,M}\alpha.$$

In particular, for any $t \in T^{M,+}$, $(\langle \kappa, \alpha^\vee \rangle + 1)\langle -w_{0,M}\alpha, v(t) \rangle \geq 0$. This means that Θ_α increases the slopes.

Theorem 6.10 ([Jon11], thm. 26, [Urb11], sect. 3.3.9)**.** *There is an exact sequence of $(M_1, T^{M,+})$-representations*

$$0 \rightarrow V_\kappa \otimes V_{\mathbf{1}}^{sm} \rightarrow V_\kappa^{lan}(w_{0,M}\kappa) \rightarrow \bigoplus_{w \in W_M^{(1)}} V_{w\cdot\kappa}^{lan}(w_{0,M}(w \cdot \kappa)) \rightarrow \cdots \rightarrow$$

$$\bigoplus_{w \in W_M^{(i)}} V_{w\cdot\kappa}^{lan}(w_{0,M}(w \cdot \kappa)) \rightarrow \cdots \rightarrow V_{w_{0,M}\cdot\kappa}^{lan}(w_{0,M}(w_{0,M} \cdot \kappa)) \rightarrow 0$$

where the first map $V_\kappa \otimes V_{\mathbf{1}}^{sm} \rightarrow V_\kappa^{lan}(w_{0,M}\kappa)$ is the natural inclusion, the second map $V_\kappa^{lan}(w_{0,M}\kappa) \rightarrow \bigoplus_{w \in W_M^{(1)}} V_{w\cdot\kappa}^{lan}(w_{0,M}(w \cdot \kappa))$ is a linear combination of the maps Θ_α for $\alpha \in \Delta_M$, and more generally the differentials $\oplus_{w \in W_M^{(i)}} V_{w\cdot\kappa}^{lan}(w_{0,M}(w \cdot \kappa)) \rightarrow \oplus_{w \in W_M^{(i+1)}} V_{w\cdot\kappa}^{lan}(w_{0,M}(w \cdot \kappa))$ are linear combinations of maps of the form $X * \cdot$ for suitable elements $X \in \mathcal{U}(\mathfrak{g})$.

Definition 6.11. *Let $\kappa \in X_\star(T)^{M,+}$. We say that a $T^{M,+}$- eigensystem λ in V_κ^{lan} is of M-small slope (abreviated $+, ss_M$) if*

$$v(\lambda) < -(\langle \kappa, \alpha^\vee \rangle + 1)w_{0,M}\alpha$$

for some $\alpha \in \Delta_M$.

We say that a $T^{M,+}$- eigensystem λ in V_κ is of M-small slope (abreviated $+, ss_M$) if

$$v(\lambda) < w_{0,M}\kappa - (\langle \kappa, \alpha^\vee \rangle + 1)w_{0,M}\alpha$$

for some $\alpha \in \Delta_M$.

Corollary 6.12. *The map $V_\kappa^{+,ss_M} \to V_\kappa^{lan,+,ss_M}(w_{0,M}\kappa)$ is an isomorphism of $(M_1, T^{M,+})$-modules.*

Proof. It follows from theorem 6.10 and lemma 6.4, that the map $V_\kappa \otimes V_{\mathbf{1}}^{sm} \to V_\kappa^{lan}(w_{0,M}\kappa)$ is an isomorphism on the small slope part. On the other hand, the map $V_\kappa \to V_\kappa \otimes V_{\mathbf{1}}$ is an isomorphism on the finite slope part. □

Example 6.13. Let $M = \mathrm{SL}_2/\mathbb{Q}_p$, with diagonal torus T and upper triangular Borel B. Then $X^\star(T) = \mathbb{Z}$, and there is a unique simple root $\alpha = 2$. For any $k \in X_\star(T)^{M,+} = \mathbb{Z}_{\geq 0}$, we have $V_k = \mathrm{Sym}^k(\mathrm{St})$. The valuation of the eigenvalues of $t = \mathrm{diag}(p, p^{-1})$ on V_k are $-k, -k+2, \cdots, k$. The $M-ss$ condition on V_k translates into the condition that the eigenvalues of t have valuation $< -k+2(k+1) = k+2$. This condition is always satisfied (in the case of SL_2) and we have $V_k = V_k^{M-ss}$. The space V_k^{lan} identifies with the space of locally analytic functions on $p\mathbb{Z}_p$ and the action of t is given by $f(z) \mapsto f(p^2 z)$. A basis of finite slope vectors in V_k^{lan} is given by the monomial functions $z \mapsto z^n$ for $n \in \mathbb{Z}_{\geq 0}$. The slopes of t on this basis are $0, 2, \cdots, 2n, \cdots$. The inclusion $V_k \hookrightarrow V_k^{lan}$ identifies V_k with the polynomial functions of degree $\leq k$, which is indeed the space defined by the slope condition $M - ss$.

6.1.7. *Distributions.* Let $\kappa_A : w_{0,M}^{-1} T(\mathbb{Z}_p) w_{0,M} \to A^\times$ be n_0-analytic. For all $n \geq n_0$, we now define $D_{\kappa_A}^{n-an} = (V_{\kappa_A}^{n-an})^\vee$ as the continuous A-dual. This is a Banach A-module, and we let $D_{\kappa_A}^{n-an,+}$ be the continuous A^+-dual of $V_{\kappa_A}^{n-an,+}$. It is an open and bounded submodule of $D_{\kappa_A}^{n-an}$. We let $D_{\kappa_A}^{lan} = \lim_n D_{\kappa_A}^{n-an} = (V_{\kappa_A}^{lan})^\vee$. This is a compact projective limit of Banach A-modules (the distributions of weight κ_A). There is a perfect pairing:

$$\langle -, - \rangle : V_{\kappa_A}^{lan} \times D_{\kappa_A}^{lan} \to A.$$

The space $D_{\kappa_A}^{lan}$ carries a right action of $(M_1, T^{M,+})$ defined by $\langle mf, \mu \rangle = \langle f, \mu m \rangle$, $\langle tf, \mu \rangle = \langle f, \mu t \rangle$ for $(t, m, f, \mu) \in T^{M,+} \times M_1 \times V_{\kappa_A}^{lan} \times D_{\kappa_A}^{lan}$. The space $D_{\kappa_A}^{lan}$ therefore carries a left action of $(M_1, T^{M,-})$ defined by $\langle m^{-1} f, \mu \rangle = \langle f, m\mu \rangle$, $\langle t^{-1} f, \mu \rangle = \langle f, t\mu \rangle$ for $(t, m, f, \mu) \in T^{M,-} \times M_1 \times V_{\kappa_A}^{lan} \times D_{\kappa_A}^{lan}$. The action of $T^{M,--}$ is by compact operators on $D_{\kappa_A}^{lan}$.

Let $\kappa \in X^\star(T)^+$. By dualizing the exact sequence of theorem 6.10, we get the following complex of $(M_1, T^{M,-})$-representations:

$$0 \to D_{w_{0,M}\cdot\kappa}^{lan}(-w_{0,M}(w_{0,M} \cdot \kappa)) \to \cdots \to \oplus_{w \in W_M^{(i)}} D_{w\cdot\kappa}^{lan}(-w_{0,M}(w \cdot \kappa))$$
$$\to \cdots \to \oplus_{w \in W_M^{(1)}} D_{w\cdot\kappa}^{lan}(-w_{0,M}(w \cdot \kappa)) \to D_\kappa^{lan}(-w_{0,M}\kappa) \to 0$$

This complex is exact except in the last degree. The cokernel of $\oplus_{w \in W_M^{(1)}} D_{w\cdot\kappa}^{lan}(-w_{0,M}(w\cdot \kappa)) \to D_\kappa^{lan}(-w_{0,M}\kappa)$ maps to $V_\kappa^\vee$. Passing to the finite slope part gives an exact sequence:

$$0 \to D_{w_{0,M}\cdot\kappa}^{lan}(-w_{0,M}(w_{0,M} \cdot \kappa))^{fs} \to \cdots \to \oplus_{w \in W_M^{(i)}} D_{w\cdot\kappa}^{lan}(-w_{0,M}(w \cdot \kappa))^{fs}$$
$$\to \cdots \to \oplus_{w \in W_M^{(1)}} D_{w\cdot\kappa}^{lan}(-w_{0,M}(w \cdot \kappa))^{fs} \to D_\kappa^{lan}(-w_{0,M}\kappa)^{fs} \to V_\kappa^\vee \to 0$$

Definition 6.14. *Let $\kappa \in X_\star(T)^{M,+}$. We say that a $T^{M,-}$- eigensystem λ in D_κ^{lan} is of M-small slope (abbreviated $-, ss_M$) if*

$$v(\lambda) > (\langle \kappa, \alpha^\vee \rangle + 1) w_{0,M}\alpha$$

for some $\alpha \in \Delta_M$.

We say that a $T^{M,-}$*- eigensystem* λ *in* $V_\kappa^\vee$ *is of* M*-small slope (abbreviated* $-, ss_M$*) if*

$$v(\lambda) > -w_{0,M}\kappa + (\langle \kappa, \alpha^\vee\rangle + 1)w_{0,M}\alpha$$

for some $\alpha \in \Delta_M$.

We have the following control theorem:

Corollary 6.15. *The map* $(D^{lan})_\kappa^{-,ss_M}(-w_{0,M}\kappa) \to (V_\kappa^\vee)^{-,ss_M}$ *is an isomorphism of* $(M_1, T^{M,-})$*-modules.*

Proof. This is the dual of corollary 6.12. □

6.2. **p-adic families of sheaves.** We will now use the locally analytic BGG resolution in families.

6.2.1. *Definition of the sheaves.* We let $K_p = K_{p,m',0}$ with $m' > 0$. Let $w \in {}^MW$. By the results of section 4.6.3, for any $n \geq 0$, over $(\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p)$ the torsor $\mathcal{M}_{dR}^{an}$ has a reduction of structure group to a torsor $\mathcal{M}_{HT,n,K_p}$ under the group $K_{p,w,M_\mu}\mathcal{M}_{\mu,n}$.

The group K_{p,w,M_μ} has an Iwahori decomposition by proposition 4.73. Moreover, $K_{p,w,M_\mu} \cap T = wT(\mathbb{Z}_p)w^{-1}$.

Let (A, A^+) be a Tate algebra over $(F, \mathcal{O}_F)$. Let $\nu_A : T^c(\mathbb{Z}_p) \to A^\times$ be an n-analytic character. Let $\kappa_A : w_{0,M}wT^c(\mathbb{Z}_p)(w_{0,M}w)^{-1} \to A^\times$ be given by $\kappa_A = -w_{0,M}w\nu_A - (w_{0,M}w\rho + \rho)$.

We can construct a sheaf $\mathcal{V}_{\nu_A}^{n-an}$ over $(\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p)$, modeled on $V_{\kappa_A}^{n-an}$. Namely consider the torsor $\pi : \mathcal{M}_{HT,n,K_p} \to (\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p)$ and $\pi \times 1 : \mathcal{M}_{HT,n,K_p} \times \mathrm{Spa}(A, A^+) \to (\pi_{HT}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p) \times \mathrm{Spa}(A, A^+)$. We let $\mathcal{V}_{\nu_A}^{n-an}$ be the subsheaf of $(\pi \times 1)_\star \mathscr{O}_{\mathcal{M}_{HT,n,K_p} \times \mathrm{Spa}(A,A^+)}$ of sections which satisfy $f(mb) = -w_{0,M_\mu}\kappa_A(b)f(m)$ for all $b \in \mathcal{B} \cap (K_{p,w,M_\mu}\mathcal{M}_{\mu,n}^1)$. We let $\mathcal{D}_{\nu_A}^{n-an} = (\mathcal{V}_{\nu_A}^{n-an})^\vee \otimes \mathcal{V}_{-2\rho_{nc}} = (\mathcal{V}_{\nu_A - 2w^{-1}\rho_{nc}}^{n-an})^\vee$, a sheaf locally modeled on $D_{\kappa_A + 2\rho_{nc}}^{an}$.

If $K_p' = K_{p,m',b}$ for $m' \geq b \geq 0$ and $m' > 0$, we have a map $(\pi_{HT,K_p'}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p') \to (\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p)$ and we obtain sheaves $\mathcal{V}_{\nu_A}^{n-an}$ and $\mathcal{D}_{\nu_A}^{n-an}$ over $(\pi_{HT,K_p'}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p')$ by pull back.

6.2.2. *First properties.* We prove that the interpolation sheaves are projective Banach sheaves.

Proposition 6.16. *The sheaves* $\mathcal{V}_{\nu_A}^{n-an}$ *and* $\mathcal{D}_{\nu_A}^{n-an}$ *are projective Banach sheaves over* $(\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p)$*. More precisely, for any affinoid* $\mathcal{U} = \mathrm{Spa}(R, R^+) \hookrightarrow (\pi_{HT,K_p}^{tor})^{-1}(]C_{w,k}[_{n,n}K_p)$*,* $\mathcal{V}_{\nu_A}^{n-an}(\mathcal{U})$ *and* $\mathcal{D}_{\nu_A}^{n-an}(\mathcal{U})$ *are projective Banach* $R\hat{\otimes}_F A$*-modules and the maps* $\mathcal{V}_{\nu_A}^{n-an}(\mathcal{U})\hat{\otimes}_R\mathscr{O}_\mathcal{U} \to \mathcal{V}_{\nu_A}^{n-an}|_\mathcal{U}$ *and* $\mathcal{D}_{\nu_A}^{n-an}(\mathcal{U})\hat{\otimes}_R\mathscr{O}_\mathcal{U} \to \mathcal{D}_{\nu_A}^{n-an}|_\mathcal{U}$ *are isomorphisms.*

Proof. It follows from proposition 4.78, that there is a finite flat morphism $f : \mathcal{U}' = \mathrm{Spa}(R', (R')^+) \to \mathcal{U}$, a finite group H acting on $\mathcal{U}'$ such that $\mathcal{U}'/H = \mathcal{U}$, such that the torsor $\mathcal{M}_{HT,n,K_p}|_{\mathcal{U}'}$ is trivial. It follows that $\mathcal{V}_{\nu_A}^{n-an}(\mathcal{U}) = (V_{\kappa_A}^{n-an}\hat{\otimes}R')^H$ (where H acts diagonally) is a direct factor of the projective $A\hat{\otimes}_F R$-module $V_{\kappa_A}^{n-an}\hat{\otimes}R'$. Similarly,

$$\mathcal{V}_{\nu_A}^{n-an}|_\mathcal{U} = (V_{\kappa_A}^{n-an} \otimes_F \mathscr{O}_{\mathcal{U}'})^H = (V_{\kappa_A}^{n-an} \otimes_F R' \otimes_R \mathscr{O}_\mathcal{U})^H = \mathcal{V}_{\nu_A}^{n-an}(\mathcal{U})\hat{\otimes}_R\mathscr{O}_\mathcal{U}.$$

The case of the distribution sheaf is similar and left to the reader. □

Corollary 6.17. *Let $U \subseteq (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$ be an open subset which is a finite union of quasi-Stein opens. Then*

$$\mathrm{R}\Gamma_{et}(U,\mathcal{V}^{n-an}_{\nu_A}) = \mathrm{R}\Gamma_{an}(U,\mathcal{V}^{n-an}_{\nu_A})$$

is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$ *and*

$$\mathrm{R}\Gamma_{et}(U,\mathcal{D}^{n-an}_{\nu_A}) = \mathrm{R}\Gamma_{an}(U,\mathcal{D}^{n-an}_{\nu_A})$$

is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$.

Proof. This follows from proposition 6.16 and lemma 2.21 (or a slight elaboration of it). □

6.2.3. *Interpolation sheaves and locally algebraic weights.* Let $\nu \in X^\star(T^c)$ and let $\kappa = -w_{0,M}w\nu - (w_{0,M}w\rho + \rho)$ be algebraic characters of T. By restriction they also define characters $\nu : T(\mathbb{Z}_p) \to F^\times$, $\kappa : w_{0,M}wT(\mathbb{Z}_p)(w_{0,M}w)^{-1} \to F^\times$.

Proposition 6.18. *Suppose $\kappa \in X^\star(T^c)^{M_\mu,+}$. Let $K_p = K_{p,m',b}$. Over $(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$ we have morphisms $\mathcal{V}_\kappa \to \mathcal{V}^{n-an}_\nu$ and $\mathcal{D}^{n-an}_\nu \to \mathcal{V}^\vee_{\kappa+2\rho_{nc}} = \mathcal{V}_{-w_{0,M}\kappa-2\rho_{nc}}$.*

Proof. This follows from the construction. Compare with section 6.1.5. □

We would like to get a similar formula for locally algebraic dominant weights. Let $m' \geq b \geq 0$. Recall that the map $\mathcal{S}^{tor}_{K^pK_{p,m',b},\Sigma} \to \mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}$ is an étale cover with group $T(\mathbb{Z}_p)/T_b(\mathbb{Z}_p)$. For any character $\chi : T(\mathbb{Z}_p)/T_b(\mathbb{Z}_p) \to F^\times$, we get an invertible sheaf $\mathscr{O}_{\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}}(\chi)$. For any sheaf of $\mathscr{O}_{\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}}$-module $\mathscr{F}$, we denote by $\mathscr{F}(\chi) = \mathscr{F} \otimes_{\mathscr{O}_{\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}}} \mathscr{O}_{\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}}(\chi)$.

Proposition 6.19. *Let $K_p = K_{p,m',0}$ with $m' > 0$. Let $m' \geq b \geq 0$. Let $n \geq b$. Let $\chi : T(\mathbb{Z}_p)/T_b(\mathbb{Z}_p) \to F^\times$ be a finite order character. Let ν_A be an n-analytic character. Over $(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$ we have that $\mathcal{V}^{n-an}_{\nu_A\chi} = \mathcal{V}^{n-an}_{\nu_A}(\chi)$.*

Proof. Let $K_p = K_{p,m',0}$ with $m' \in \mathbb{Z}_{>0}$. Let $K'_p = K_{p,m',b}$ with $m' \geq b$. The map $(\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{n,n}K'_p) \to (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$ is an étale cover of group $T(\mathbb{Z}_p)/T_b(\mathbb{Z}_p)$ since $]C_{w,k}[_{n,n}K'_p =]C_{w,k}[_{n,n}K_p$. We also have a map of torsors $\mathcal{M}_{HT,m,n,K'_p} \to \mathcal{M}_{HT,m,n,K_p}$ equivariant for the map: $K'_{p,w,M_\mu}\mathcal{M}^1_{\mu,n,n} \to K_{p,w,M_\mu}\mathcal{M}^1_{\mu,n,n}$. We form the quotient:

$$K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n}/K'_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n} = wT(\mathbb{Z}_p)w^{-1}/wT_b(\mathbb{Z}_p)w^{-1}.$$

By taking the pushout of the map $\mathcal{M}_{HT,m,n,K'_p} \to \mathcal{M}_{HT,m,n,K_p}$ via $K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n} \to wT(\mathbb{Z}_p)w^{-1}/wT_b(\mathbb{Z}_p)w^{-1}$, we get a map:

$$(\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{m,n}K'_p) \to \mathcal{M}_{HT,m,n,K_p} \times_{K_{p,w,M_\mu}\mathcal{M}^1_{\mu,m,n}} (wT(\mathbb{Z}_p)w^{-1}/wT_b(\mathbb{Z}_p)w^{-1})$$

This map is necessarily an isomorphism, because the left hand side is an étale cover of $(\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)$ of group $T(\mathbb{Z}_p)/T_b(\mathbb{Z}_p)$ and the right hand side is an étale cover of group $wT(\mathbb{Z}_p)w^{-1}/wT_b(\mathbb{Z}_p)w^{-1}$. Moreover, the map is equivariant under the isomorphism $T(\mathbb{Z}_p)/T_b(\mathbb{Z}_p) \to wT(\mathbb{Z}_p)w^{-1}/wT_b(\mathbb{Z}_p)w^{-1}$ given by conjugation by w. It follows that the étale cover $(\pi^{tor}_{HT,K'_p})^{-1}(]C_{w,k}[_{m,n}K'_p) \to (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{m,n}K_p)$ can be realized as a pushout of the torsor $\mathcal{M}_{HT,n,K_p}$.

We deduce from lemma 6.6 that $V^{n-an}_{\kappa_A} \otimes w\chi = V_{\kappa_A \otimes (\omega_{0,M} w \chi^{-1})}$. The lemma follows. $\square$

Corollary 6.20. *Let $\nu = \nu_{alg}\chi$ is a locally algebraic character of $T^c(\mathbb{Z}_p)$, with $\nu_{alg} \in X^*(T)$ algebraic and $\chi : T(\mathbb{Z}_p)/T_b(\mathbb{Z}_p) \to F^\times$ a finite order character then if $\kappa_{alg} = -w_{0,M} w \nu_{alg} + (w_{0,M} w\rho - \rho)$ is M-dominant, we have maps of sheaves $\mathcal{V}_{\kappa_{alg}}(\chi) \to \mathcal{V}^{n-an}_{\nu}$ and $\mathcal{D}^{n-an}_{\nu} \to \mathcal{V}_{-2\rho_{nc} - \kappa_{alg}}(-\chi)$.*

Proof. This is a combination of propositions 6.18 and 6.19. $\square$

6.2.4. *Definition of the action of the Hecke algebra.* Let $t \in T^+$. Let $K_p = K_{p,m',0}$. We consider the Hecke correspondence:

$$\begin{array}{ccc} & S^{tor}_{K^p(K_p \cap tK_pt^{-1}),\Sigma''} & \\ \swarrow^{p_2} & & \searrow^{p_1} \\ S^{tor}_{K^pK_p,\Sigma} & & S^{tor}_{K^pK_p,\Sigma'} \end{array}$$

Over $p_2^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)\big) \cap p_1^{-1}\big((\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)\big)$ we have a map

$$\begin{array}{ccc} p_1^\star \mathcal{M}^{an}_{HT} & \xrightarrow{[t^{-1}]} & p_2^\star \mathcal{M}^{an}_{HT} \\ \uparrow & & \uparrow \\ p_1^\star \mathcal{M}_{HT,n,K_p} & & p_2^\star \mathcal{M}_{HT,n,K_p} \end{array}$$

which is represented by $K_{p,w,M_\mu}\mathcal{M}_n w t^{-1} w^{-1} K_{p,w,M_\mu}\mathcal{M}_n$ by proposition 4.81. So far, all this discussion depends only on the double class $[K_ptK_p]$ and therefore only on the image of $t \in T^+/T(\mathbb{Z}_p)$. In particular, if $t \in T(\mathbb{Z}_p)$, all the maps are the identity.

We now consider the following map:

$$\widetilde{[t^{-1}]} : p_1^\star \mathcal{M}^{an}_{HT}/\mathcal{U}^{an} \to p_2^\star \mathcal{M}^{an}_{HT}/\mathcal{U}^{an}$$

which is given by $x\mathcal{U}^{an} \mapsto [t^{-1}]xwtw^{-1}\mathcal{U}^{an}$. This map depends on t and not only on its class in $T^+/T(\mathbb{Z}_p)$.

Lemma 6.21. (1) *The map $\widetilde{[t^{-1}]}$ induces a map*

$$p_1^\star \mathcal{M}_{HT,n,K_p}/\mathcal{U}^{an} \to p_2^\star \mathcal{M}_{HT,n,K_p}/\mathcal{U}^{an}.$$

(2) *If $t \in T^{++}$ and $n \geq 1$, this map factors through*

$$p_1^\star \mathcal{M}_{HT,n,K_p}/\mathcal{U}^{an} \hookrightarrow p_1^\star \mathcal{M}_{HT,n-1,K_p}/\mathcal{U}^{an} \to p_2^\star \mathcal{M}_{HT,n,K_p}/\mathcal{U}^{an}.$$

(3) *Let ν_A be an n-analytic weight. Then $\widetilde{[t^{-1}]}$ induces a compact morphism (in the sense of definition 2.8):*

$$\widetilde{[t^{-1}]} : p_2^\star \mathcal{V}^{n-an}_{\nu_A} \to p_1^\star \mathcal{V}^{n-an}_{\nu_A}$$

which is locally modeled on the morphism $wtw^{-1} : V^{n-an}_{\kappa_A} \to V^{n-an}_{\kappa_A}$ defined in section 6.1.3.

Proof. Easy and left to the reader. Observe that $wtw^{-1} \in T^{M_\mu,+}$. $\square$

We can now attach to $t \in T^+$ the following normalized map that will be used to obtain an action on cohomology :

$$t : \mathrm{R}(p_1)_\star p_2^\star \mathcal{V}_{\nu_A}^{n-an} \to \mathcal{V}_{\nu_A}^{n-an}$$

and wich is the composite of $\widetilde{[t^{-1}]} : \mathrm{R}(p_1)_\star p_2^\star \mathcal{V}_{\nu_A}^{n-an} \to \mathrm{R}(p_1)_\star p_1^\star \mathcal{V}_{\nu_A}^{n-an}$ and $\langle -w^{-1}w_{0,M}\rho + \rho, t\rangle \mathrm{Tr}_{p_1} : \mathrm{R}(p_1)_\star p_1^\star \mathcal{V}_{\nu_A}^{n-an} \to \mathcal{V}_{\nu_A}^{n-an}$.

Lemma 6.22. *If $t \in T(\mathbb{Z}_p)$, then*

$$t : \mathrm{R}(p_1)_\star p_2^\star \mathcal{V}_{\nu_A}^{n-an} = \mathcal{V}_{\nu_A}^{n-an} \to \mathcal{V}_{\nu_A}^{n-an}$$

acts via scalar multiplication by $\nu_A(t)$.

Proof. This follows from the identity $\nu_A = w^{-1}w_{0,M}\kappa_A - w^{-1}w_{0,M}\rho + \rho$. The scalar multiplication by $w^{-1}w_{0,M}\kappa_A(t)$ comes from the map $\widetilde{[t^{-1}]}$, the multiplication by $\langle -w^{-1}w_{0,M}\rho + \rho, t\rangle$ comes from the trace map. □

By duality we also obtain a morphism: $p_2^\star \mathcal{D}_{\nu_A}^{n-an} \to p_1^\star \mathcal{D}_{\nu_A}^{n-an}$ which is locally modeled on $(wtw^{-1})^{-1} : D_{\kappa_A + 2\rho_{nc}}^{n-an} \to D_{\kappa_A+2\rho_{nc}}^{n-an}$.

6.3. **Locally analytic overconvergent cohomology.** Let $w \in {}^MW$. For a choice of $+$ or $-$ and a weight $\nu_A : T(\mathbb{Z}_p) \to A^\times$ we want to define a finite slope overconvergent, locally analytic cohomology $\mathrm{R}\Gamma_{w,an}(K^p, \nu_A)^{\pm, fs}$ and the cuspidal counterpart $\mathrm{R}\Gamma_{w,an}(K^p, \nu_A, cusp)^{\pm, fs}$ by taking cohomologies of the analytic sheaves $\mathcal{V}_{\nu_A}^{n-an}$ or $\mathcal{D}_{\nu_A}^{n-an}$ (for n large enough), with suitable support conditions of neighborhoods of the inverse image of $\mathcal{P}_\mu \backslash \mathcal{P}_\mu w K_p$ by the Hodge-Tate period map, and applying a finite slope projector.

6.3.1. *Relative spectral theory.* We recall briefly the relative spectral theory for compact operators. The original reference is [Col97]. Let (A, A^+) be a noetherian complete Tate $(F, \mathcal{O}_F)$-algebra. Let $M^\bullet \in Ob(\mathcal{K}^{proj}(A))$ and let $T \in \mathrm{End}_{\mathcal{D}(\mathbf{Ban}(A))}(M^\bullet)$ be a compact operator.

We begin by describing a non-canonical construction which depends on a lift $\tilde{T} \in \mathrm{End}_A(M^\bullet)$ which is compact in all degrees. We let

$$\tilde{P}(X) = \prod_k \det(1 - X\tilde{T}|M^k) \in A[[X]]$$

be the (total) Fredholm determinant of $\tilde{T}$ (see [Col97], section A 2). The series $\tilde{P}(X)$ is an entire series and we let $\tilde{\mathcal{Z}} \hookrightarrow \mathbb{G}_m^{an} \times \mathrm{Spa}(A, A^+)$ be the vanishing locus of $\tilde{P}$. This is the spectral variety associated to $\tilde{T}$. The morphism $\pi : \tilde{\mathcal{Z}} \to \mathrm{Spa}(A, A^+)$ is locally quasi-finite, flat and partially proper (see [AIP18], thm. B1 for a short proof in the language of adic spaces). Any point $z \in \tilde{\mathcal{Z}}$ has a neighborhood $\mathcal{U}_z$ such that $\overline{z} \subseteq \mathcal{U}_z$ and $\mathcal{U}_z \to \pi(\mathcal{U}_z)$ is finite flat. We can describe more precisely a neighborhood $\mathcal{U}_z$ of z. Let $x \in \mathrm{Spa}(A, A^+)$ be the image of z. Then there is a neighborhood $\mathrm{Spa}(B, B^+)$ of x in $\mathrm{Spa}(A, A^+)$ and a factorization in $B[[X]]$: $\tilde{P}(X) = \tilde{R}(X)\tilde{Q}(X)$ where $\tilde{R}(X)$ is a Fredholm series, $\tilde{Q}(X) = 1 + a_1X + \cdots + a_dX^d$ is a polynomial with $a_d \in B^\times$, $\tilde{R}(X)$ and $\tilde{Q}(X)$ are prime to each other and $\tilde{Q}(z) = 0$. Moreover, $\mathcal{U}_z = V(\tilde{Q}(X)) \subseteq \mathbb{A}^1 \times \mathrm{Spa}(B, B^+)$ is a neighborhood of z in $\tilde{\mathcal{Z}}$.

Over $\tilde{\mathcal{Z}}$ we have a complex of coherent sheaves $\tilde{\mathcal{M}}^{\bullet, fs}$ whose definition we briefly recall. Let $\tilde{Q}^*(X) = X^d\tilde{Q}(X^{-1})$. Over $\mathrm{Spa}(B, B^+)$ we have a unique decomposition $M^\bullet \hat{\otimes}_A B = M^\bullet(\tilde{Q}) \oplus N^\bullet(\tilde{Q})$ where $\tilde{Q}^*(\tilde{T})$ is zero on $M^\bullet(\tilde{Q})$ and invertible on

$N^\bullet(\tilde{Q})$. It follows that $M^\bullet(\tilde{Q})$ has a natural structure of complex of $B[X]/\tilde{Q}(X)$-module of finite type (with X^{-1} acting like $\tilde{T}$) and we let $\tilde{\mathcal{M}}^{\bullet,fs}|_{V(\tilde{Q}(X))} = M^\bullet(\tilde{Q})$. These glue to give the complex $\tilde{\mathcal{M}}^{\bullet,fs}$ over $\tilde{\mathcal{Z}}$. We observe that by construction $\tilde{\mathcal{M}}^{\bullet,fs}$ is a perfect complex of $\pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)}$-modules. Moreover, if $M^\bullet$ is concentrated in the range $[a,b]$, so is $\tilde{\mathcal{M}}^{\bullet,fs}$.

We observe at this point that neither $\tilde{\mathcal{M}}^\bullet$, nor $\tilde{\mathcal{Z}}$ are canonical objects. They depend on the choice of $\tilde{T}$. We now introduce some objects that do not depend on the choice of $\tilde{T}$.

Let $\mathcal{M}^\bullet$ be the complex of projective Banach sheaves over $\mathrm{Spa}(A,A^+)$ attached to $M^\bullet$. We define $\mathcal{M}^{\bullet,fs} = \pi_\star\tilde{\mathcal{M}}^{\bullet,fs}$. It follows from the construction that there is a natural map $\mathcal{M}^{\bullet,fs} \to \mathcal{M}^\bullet$. There is also a section $\mathcal{M}^\bullet \to \mathcal{M}^{\bullet,fs}$. By adjunction, this section is provided by the map $\pi^\star\mathcal{M}^\bullet \to \tilde{\mathcal{M}}^{\bullet,fs}$ (which in the above notations, is locally given by the projection $M^\bullet\hat{\otimes}B \to M^\bullet(\tilde{Q})$, orthogonal to $N^\bullet(\tilde{Q})$).

The complex $\mathcal{M}^{\bullet,fs}$ is thus a direct factor of $\mathcal{M}^\bullet$. For any $z \in \mathrm{Spa}(A,A^+)$, we have that $\mathcal{M}^{\bullet,fs} \otimes^L_{\mathscr{O}_{\mathrm{Spa}(A,A^+)}} k(z) = (M^\bullet \otimes^L_A k(z))^{fs}$ where the second sup script fs is taken in the sense of section 5.2.2.

Any other lift of T will produce another complex $\mathcal{M}^{',\bullet,fs}$, canonically quasi-isomorphic to $\mathcal{M}^{\bullet,fs}$. Thus the complex $\mathcal{M}^{\bullet,fs}$ viewed as an object of the derived category of abelian sheaves over $\mathrm{Spa}(A,A^+)$ depends on $\tilde{T}$ up to a unique isomorphism. Therefore, $\mathcal{M}^{\bullet,fs}$ is canonically attached to T and we call it the finite slope part of $M^\bullet$. The choice of $\tilde{T}$ allows us to construct the perfect complex $\tilde{\mathcal{M}}^{\bullet,fs}$ which is some sort of nice "resolution" of $\mathcal{M}^{\bullet,fs}$ over the spectral variety $\tilde{\mathcal{Z}}$.

We let $\mathrm{H}^\bullet(\tilde{\mathcal{M}}^{\bullet,fs})$ be the associated graded module over $\tilde{\mathcal{Z}}$, obtained by taking cohomology. We also let $\mathcal{Z}$ be the closed subspace of $\tilde{\mathcal{Z}}$, equal to the support of $\mathrm{H}^\bullet(\tilde{\mathcal{M}}^{\bullet,fs})$. This is also independent of $\tilde{T}$ and is the spectral variety associated to T. Finally, we note that $\mathrm{H}^\bullet(\mathcal{M}^{\bullet,fs}) = \pi_\star\mathrm{H}^\bullet(\tilde{\mathcal{M}}^{\bullet,fs})$.

6.3.2. *Relative spectral theory for an algebra of operators.* We now let $\mathbb{Z}^r_{\geq 0}$ be the free monoid on r generators, generated by element $T_1, \cdots, T_r$. We assume that we have an action of $\mathbb{Z}^r_{\geq 0}$ on an object $M^\bullet \in Ob(\mathcal{K}^{proj}(A))$. We also assume that the operators $T \in \mathbb{Z}^r_{\geq 1}$ are potent compact.

For any choice of $T \in \mathbb{Z}^r_{\geq 1}$ acting compactly, and any lift $\tilde{T}$ of T to a compact operator on the complex $M^\bullet$, we can construct a spectral variety $\tilde{\mathcal{Z}}_T$, and a complex of sheaves $\tilde{\mathcal{M}}^{\bullet,fs}_T$ over $\tilde{Z}_T$. The complex of sheaves $\mathcal{M}^{\bullet,fs}_T$ on $\mathrm{Spa}(A,A^+)$ (the pushforward of $\tilde{\mathcal{M}}^{\bullet,fs}_T$) is easily shown to be independent of the particular choice of T (compare with lemma 5.3), and is called the finite slope part of $\mathcal{M}^\bullet$, and denoted $\mathcal{M}^{\bullet,fs}$.

Let us fix again a compact operator $T \in \mathbb{Z}^r_{\geq 1}$. Let $\mathcal{Z}_T \hookrightarrow \mathbb{G}_m \times \mathrm{Spa}(A,A^+)$ be the spectral variety associated to T. We let $\mathscr{O}_\mathcal{Z}$ be the coherent $\mathscr{O}_{\mathcal{Z}_T}$-algebra generated by the image of $T_1, \cdots, T_r$ in $\oplus_k \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,fs}_T)$. Let $\mathcal{Z}$ be the associated adic space, finite over $\mathcal{Z}_T$. We call this space the spectral variety associated to $T_1, \cdots, T_r$. There is a structural morphism $\mathcal{Z} \to \mathrm{Spa}(A,A^+)$ (the weight morphism). There is also a morphism $\mathcal{Z} \to (\mathbb{G}_m^{an})^r$ given by $T_1^{-1}, \cdots, T_r^{-1}$ (as one checks that since T is invertible, so are each of the T_i), and we have an embedding $\mathcal{Z} \hookrightarrow (\mathbb{G}_m^{an})^r \times \mathrm{Spa}(A,A^+)$. Finally, the space $\mathcal{Z}$ carries a graded sheaf of modules $\mathrm{H}^\bullet(\tilde{\mathcal{M}}^{\bullet,fs})$ (with pushforward along the finite map $\mathcal{Z} \to \mathcal{Z}_T$ equal to $\mathrm{H}^\bullet(\tilde{\mathcal{M}}^{\bullet,fs}_T)$).

6.3.3. *The case of projective limits.* We claim that all the material of the last sections applies to the more general case where the complex $M^\bullet \in Ob(\mathcal{K}^{proj}(A))$ is replaced by an object "$\lim_i$" $M_i^\bullet \in Ob(\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A)))$. This was explained in section 5.2.4 when the base if a field $(F, \mathcal{O}_F)$,the key observation being lemma 2.6 which reduces the theory of compact operators on objects "$\lim_i$" $M_i^\bullet \in Ob(\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A)))$ to theory of compact operators on objects $M^\bullet \in Ob(\mathcal{K}^{proj}(A))$. Details are left to the reader.

6.3.4. *First definition.* Let $K_p = K_{p,m',0}$ with $m' > n$ and let $\nu_A : T(\mathbb{Z}_p) \to A^\times$ be an n-analytic weight. We fix $t \in T^{++}$ and we assume that $\min(t) = \inf_{\alpha \in \Phi^+} v(\alpha(t)) \geq 1$ in order to simplify notations. We let T be the associated Hecke operator.

We have that

$$T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[))\cap(T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]Y_{w,k}[})) \subseteq (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n+1,\overline{n+1}}K_p)$$
$$\subseteq (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$$

by lemma 3.29 and therefore, the sheaves $\mathcal{V}^{n-an}_{\nu_A}$ and $\mathcal{D}^{n-an}_{\nu_A}$ are defined over a neighborhood of $T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[)) \cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]Y_{w,k}[}))$.

We also have that

$$T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap(T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[)) \subseteq (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{\overline{n+1},n+1}K_p)$$
$$\subseteq (\pi^{tor}_{HT,K_p})^{-1}(]C_{w,k}[_{n,n}K_p)$$

and therefore, the sheaves $\mathcal{V}^{n-an}_{\nu_A}$ and $\mathcal{D}^{n-an}_{\nu_A}$ are defined over a neighborhood of $T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[})) \cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))$ as well.

We define:

$$\mathrm{R}\Gamma_{w,n-an}(K^pK_p, \nu_A)^{+,fs} :=$$
$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap(T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))}(T^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}^{n-an}_{\nu_A})^{+,fs}.$$

Implicit in this definition is that it makes sense to take the finite slope part: namely the cohomology is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$ and that $\mathcal{H}^+_{p,m',0}$ acts on it in a way that $\mathcal{H}^{++}_{p,m',0}$ acts by potent compact operators. This is proved in Theorem 6.23 below. Note that $\mathrm{R}\Gamma_{w,n-an}(K^pK_p, \nu_A)^{+,fs}$ is an object of the derived category of abelian sheaves on $\mathrm{Spa}(A, A^+)$. See sections 6.3.1, 6.3.2, 6.3.3.

Similarly, we define:

$$\mathrm{R}\Gamma_{w,n-an}(K^pK_p, \nu_A)^{-,fs} :=$$
$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[))\cap(T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]Y_{w,k}[}))}((T^t)^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[), \mathcal{D}^{n-an}_{\nu_A})^{-,fs}.$$

Again implicit in this definition is that it makes sense to take the finite slope part: namely the cohomology is an object of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$ and that $\mathcal{H}^-_{p,m',0}$ acts on it in a way that $\mathcal{H}^{--}_{p,m',0}$ acts by potent compact operators. This is proved in Theorem 6.23 below.

We have similar definitions for cuspidal cohomologies $\mathrm{R}\Gamma_{w,n-an}(K^pK_p, \nu_A, cusp)^{+,fs}$ and $\mathrm{R}\Gamma_{w,n-an}(K^pK_p, \nu_A, cusp)^{-,fs}$

6.3.5. *Existence of finite slope cohomology.* We now justify that the cohomologies introduced in the previous section are well defined.

Theorem 6.23. *Let $K_p = K_{p,m',0}$ for some $m' > n$, $w \in {}^M W$, and $\nu_A : T(\mathbb{Z}_p) \to A^\times$ an n-analytic character.*

(1) *The cohomologies*

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))}(T^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}^{n-an}_{\nu_A}),$$

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]Y_{w,k}[}))}((T^t)^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[), \mathcal{D}^{n-an}_{\nu_A})$$

are objects of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$.

(2) *There is an action of $\mathcal{H}^+_{p,m',0}$ on*

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))}(T^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}^{n-an}_{\nu_A})$$

for which $\mathcal{H}^{++}_{p,m',0}$ acts via compact operators.

(3) *There is an action of $\mathcal{H}^-_{p,m',0}$ on*

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]Y_{w,k}[}))}((T^t)^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[), \mathcal{D}^{n-an}_{\nu_A})$$

for which $\mathcal{H}^{--}_{p,m',0}$ acts via compact operators.

(4) *All these statements hold also for the cuspidal cohomology.*

Proof. The property that the objects at hand are objects of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$ is corollary 6.17. The rest of the argument follows almost verbatim the proof of theorem 5.10. In particular the compactity of the operators $\in \mathcal{H}^{\pm\pm}_{p,m',0}$ follows from lemma 2.25. The details are left to the reader. □

6.3.6. *Change of analyticity radius.*

Theorem 6.24. *Let $m' > n+1$. Let ν_A be an n-analytic character. The maps $\mathrm{R}\Gamma_{w,n+1-an}(K^pK_p, \nu_a)^{+,fs} \to \mathrm{R}\Gamma_{w,n-an}(K^pK_p, \nu_A)^{+,fs}$ and $\mathrm{R}\Gamma_{w,n-an}(K^pK_p, \nu_A)^{-,fs} \to \mathrm{R}\Gamma_{w,n+1-an}(K^pK_p, \nu_A)^{-,fs}$ are quasi-isomorphisms.*

Proof. Let $T = [K_{p,m',0} t K_{p,m',0}]$ for $t \in T^{++}$ satisfying $\min(t) \geq 1$. The endomorphism T of

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))}(T^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}^{n-an}_{\nu_A})$$

factors into

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))}(T^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}^{n-an}_{\nu_A})$$

$$\longrightarrow$$

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))}(T^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}^{n+1-an}_{\nu_A})$$

$$\longrightarrow$$

$$\mathrm{R}\Gamma_{T^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(\overline{]X_{w,k}[}))\cap (T^t)^{n+1}((\pi^{tor}_{HT,K_p})^{-1}(]Y_{w,k}[))}(T^{n+1}(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,k}[), \mathcal{V}^{n-an}_{\nu_A}).$$

The $-$ case follows similarly. □

6.3.7. *Change of support condition.* It is important to us that the cohomology $\mathrm{R}\Gamma_{w,n-an}(K^pK_p,\nu_A)^{\pm,fs}$ can actually be realized as the finite slope part of cohomology groups with different support conditions. The following definition is similar to definition 5.11. We start by fixing an element $t \in T^{++}$ such that $\min(t) \geq 1$ and we set $C = \max(t)$.

Definition 6.25. *Let $m' > n$. A $(+, w, K_{p,m',0}, n-an)$-allowed support is a pair $(\mathcal{U}, \mathcal{Z})$ where:*

1. *$\mathcal{U}$ is an open subset of $\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}$ which is a finite union of quasi-Stein open subsets.*
2. *$\mathcal{Z}$ is a closed subset of $\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}$ whose complement is a finite union of quasi-Stein open subsets.*
3. *There exists $m, l, s \in \mathbb{Z}_{\geq 0}$ with $m, l \geq n+1$ such that:*

$$(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{m,\overline{0}}K_{p,m',0} \cap]C_{w,k}[_{0,\overline{l+s}}K_{p,m',0}) \subseteq \mathcal{Z} \subseteq (\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{0,C\overline{l}}K_{p,m',0}),$$

$$(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{m+s,\overline{0}}K_{p,m',0} \cap]C_{w,k}[_{0,\overline{l}}K_{p,m',0}) \subseteq \mathcal{U} \subseteq (\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{Cm,-1}K_{p,m',0}).$$

Let $m' > n$. A $(-, w, K_{p,m',0}, n-an)$-allowed support is a pair $(\mathcal{U}, \mathcal{Z})$ where:

1. *$\mathcal{U}$ is an open subset of $\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}$ which is a finite union of quasi-Stein open subsets.*
2. *$\mathcal{Z}$ is a closed subset of $\mathcal{S}^{tor}_{K^pK_{p,m',0},\Sigma}$ whose complement is a finite union of quasi-Stein open subsets.*
3. *There exists $m, l, s \in \mathbb{Z}_{\geq 0}$ with $m, l \geq n+1$ such that:*

$$(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{\overline{m+s},0}K_{p,m',0} \cap]C_{w,k}[_{\overline{0},n}K_{p,m',0}) \subseteq \mathcal{Z} \subseteq (\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{C\overline{m},0}K_{p,m',0}),$$

$$(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{\overline{m},0}K_{p,m',0} \cap]C_{w,k}[_{\overline{0},l+s}K_{p,m',0}) \subseteq \mathcal{U} \subseteq (\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{-1,Cl}K_{p,m',0}).$$

Theorem 6.26. *Let $m' > n$, $w \in {}^MW$, and ν_A an n-analytic character.*

1. *Let $(\mathcal{U}, \mathcal{Z})$ be a $(+, w, K_{p,m',0}, n-an)$-allowed support condition. Then $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}^{n-an}_{\nu_A})$ and $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}^{n-an}_{\nu_A}(-D))$ are objects of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$ and carry a canonically defined, potent compact action of T^s. Moreover there are canonical isomorphisms*
$$\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0}, \nu_A)^{+,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}^{n-an}_{\nu_A})^{T^s-fs}$$
and
$$\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0}, \nu_A, cusp)^{+,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{V}^{n-an}_{\nu_A}(-D))^{T^s-fs}.$$
2. *Let $(\mathcal{U}, \mathcal{Z})$ be a $(-, w, K_{p,m',0}, n-an)$-allowed support condition. Then $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{D}^{n-an}_{\nu_A})$ and $\mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{D}^{n-an}_{\nu_A}(-D))$ are objects of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$ and carry a canonically defined, potent compact action of T^s. Moreover there are canonical isomorphisms*
$$\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0}, \nu_A)^{-,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{D}^{n-an}_{\nu_A})^{T^s-fs}$$
and
$$\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0}, \nu_A, cusp)^{-,fs} \simeq \mathrm{R}\Gamma_{\mathcal{Z}\cap\mathcal{U}}(\mathcal{U}, \mathcal{D}^{n-an}_{\nu_A}(-D))^{T^s-fs}.$$

Proof. This is very similar to the proof of theorem 5.13, and left to the reader. □

6.3.8. *Change of level.* Now we investigate how the finite slope cohomologies $\mathrm{R}\Gamma_{w,n-an}(K^pK_p,\nu_A)^{\pm,fs}$ and $\mathrm{R}\Gamma_{w,n-an}(K^pK_p,\nu_A,cusp)^{\pm,fs}$ vary with the level K_p.

Theorem 6.27. *For all $w\in{}^MW$ and all $m''\geq m'>n$, the pullback map*

$$\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0},\nu_A)^{+,fs}\to\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m'',0},\nu_A)^{+,fs}$$

and the trace map

$$\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m'',0},\nu_A)^{-,fs}\to\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0},\nu_A)^{-,fs}$$

are quasi-isomorphisms, compatible with the action of $\mathbb{Q}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]$, and the same statements are true for cuspidal cohomology.

Proof. This is very similar to the proof of theorem 5.14. Details are left to the reader. □

As a result of the theorems 6.24 and 6.27, we can let

$$\mathrm{R}\Gamma_{w,an}(K^p,\nu_A)^{\pm,fs}\text{ and }\mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{\pm,fs}$$

denote respectively

$$\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0},\nu_A)^{\pm,fs}\text{ and }\mathrm{R}\Gamma_{w,n-an}(K^pK_{p,m',0},\nu_A,cusp)^{\pm,fs}$$

for any $m'>n$, as these spaces have been canonically identified.

6.4. **Cohomological vanishing.** We now state a cohomological vanishing conjecture. Compare with conjecture 5.20.

Conjecture 6.28. *For all $w\in{}^MW$, the cohomology $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{\pm,fs}$ is concentrated in degree $[0,\ell_\pm(w)]$ and the cohomology $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A)^{\pm,fs}$ is concentrated in degree $[\ell_\pm(w),d]$*

In this section we prove one half of this conjecture. Compare with theorem 5.18.

Theorem 6.29. *For all $w\in{}^MW$, the cohomology $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{\pm,fs}$ is concentrated in degree $[0,\ell_\pm(w)]$.*

Proof. This is exactly as the proof of theorem 5.18, granting the lemma below. □

The key lemma to prove the theorem is the following (this lemma and its proof is inspired by [AIP15]):

Lemma 6.30. *Let $K_p=K_{p,m',0}$. There exists n_0 such that for all $n\geq n_0$ the following holds. Let $\mathcal{U}\subseteq]C_{w,k}[_{n,n}K_p$ be an open affinoid. Let $\mathcal{V}$ be the inverse image of $\mathcal{S}$ in $\mathcal{S}^{tor}_{K_pK^p,\Sigma}$. Then $\mathrm{R}\Gamma((\pi^{tor}_{HT,K_p})^{-1}(\mathcal{U}),\mathcal{V}^{n-an}_{\nu_A}(-D))$ and $\mathrm{R}\Gamma((\pi^{tor}_{HT,K_p})^{-1}(\mathcal{U}),\mathcal{D}^{n-an}_{\nu_A}(-D))$ are concentrated in degree 0.*

Proof. For any $K'_p\subseteq K_p$, we let $\mathfrak{S}^{tor}_{K'_pK_p,\Sigma}$ be an integral toroidal compactification and $\mathfrak{S}^\star_{K'_pK^p}$ be an integral minimal compactification (whose existence is given by the main results of [MS11]). Let $\mathfrak{U}$ be a formal model for $\mathcal{U}$, realized as an open subset of the normalization of a blow-up of $\mathfrak{FL}_{G,\mu}$. For K'_p small enough, we have nice normal formal models $\mathfrak{S}^{\star,mod}_{K'_pK^p,\mathfrak{U}}\to\mathfrak{S}^\star_{K'_pK^p}$ for the map $\pi^{-1}_{HT,K_p}(\mathcal{U})\to\mathcal{S}^\star_{K'_pK^p}$ and $\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}\to\mathfrak{S}^{tor}_{K'_pK^p,\Sigma}$ for the map $(\pi^{tor}_{HT,K_p})^{-1}(\mathcal{U})\to\mathcal{S}^{tor}_{K'_pK^p,\Sigma}$. We have a map $f:\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}\to\mathfrak{S}^{\star,mod}_{K'_pK^p,\mathfrak{U}}$.

By proposition 4.78 for $K'_p \subseteq K_p$ small enough, the torsor $\mathcal{M}_{HT,n,K_p}$ is trivial on the generic fiber of a Zariski cover of $\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}$.

We can construct a formal model for the Banach sheaf $\mathcal{V}^{n-an}_{\nu_A}$ that we denote $\mathfrak{V}^{n-an,+}_{\nu_A}$ over $\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}$. Indeed let $\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}} = \cup_i \mathfrak{V}_{K'_p,i}$ be a Zariski covering, with the property that the torsor $\mathcal{M}_{HT,n,K_p}$ is trivial over the generic fiber $\mathcal{V}_{K'_p,i}$ of $\mathfrak{V}_{K'_p,i}$. We fix such a trivialization. We can construct the associated 1-cocycle: over each intersection $\mathfrak{V}_{K'_p,i,j} = \mathfrak{V}_{K'_p,i} \cap \mathfrak{V}_{K'_p,j}$ we have an element $m_{i,j} \in K_{p,w,M_\mu}\mathcal{M}_{\mu,n}$ describing the change of trivialization of the torsor $\mathcal{M}_{HT,n,K_p}$. Over $\mathfrak{V}_{K'_p,i}$ we let $\mathfrak{V}^{n-an,+}_{\nu_A} = V^{n-an,+}_{\kappa_A} \hat{\otimes}_{\mathcal{O}_F} \mathscr{O}_{\mathfrak{V}_{K'_p,i}}$ and we glue these sheaves using multiplication by $m_{i,j}$ on each intersection. The sheaf $\mathfrak{V}^{n-an,+}_{\nu_A}$ is a flat formal Banach sheaf (see section 2.5.2). We claim that it is also small (see again section 2.5.2). Indeed, it follows from lemma 6.3 that the representation $V^{n-an,+}_{\kappa_A} \otimes_{A^+} A^+/A^{++}$ of $K_{p,w,M_\mu}\mathcal{M}_{\mu,n}$ is an inductive limit of finite free A^+/A^{++}-submodules $\mathrm{colim}_i V_i$, stable under $K_{p,w,M_\mu}\mathcal{M}_{\mu,n}$, and such that over V_i/V_{i-1} the action factors over the quotient K^p_{p,w,M_μ} of K_{p,w,M_μ} by its maximal normal pro-p subgroup. We find that $K^p_{p,w,M_\mu} = wT(\mathbb{Z}_p)w/wT_1w$ is a finite abelian group.

It follows that the sheaf $\mathfrak{V}^{n-an,+}_{\nu_A}/A^{++} = \mathrm{colim}_i \mathscr{F}_i$ is an inductive limit of locally free sheaves of $A^+/A^{++} \otimes \mathscr{O}_{\mathfrak{V}'}$-modules with the property that $\mathscr{F}_i/\mathscr{F}_{i-1}$ runs through a finite family of torsion invertible sheaves. We now fix $\mathcal{L}$ an ample line bundle over $\mathfrak{S}^{tor}_{K'_pK^p,\mathfrak{U}}$.

We claim that there exists $m \geq 0$ such that $\mathrm{R}\Gamma(\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}, \mathcal{L}^m \otimes \mathfrak{V}^{n-an,+}_{\nu_A}(-D_{K_p}))$ (where $-D_{K_p}$ is the pull back of the boundary divisor at level K_p) is concentrated in degree 0. The cohomology is represented by a complex of A^+-modules which are completions of free A^+-modules (take the Čech complex associated with the covering $\cup \mathfrak{V}'_i$). It follows from lemma 6.31 that its suffices to prove that there exists $m \geq 0$ such that $\mathrm{R}\Gamma(\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}, \mathcal{L}^m \otimes \mathfrak{V}^{n-an,+}_{\nu_A}/A^{++}(-D_{K_p}))$ is concentrated in degree 0. We are therefore reduced to prove that there exists m such that $\mathrm{R}\Gamma(\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}, \mathcal{L}^m \otimes \mathscr{F}_i/\mathscr{F}_{i-1}(-D_{K_p}))$ is concentrated in degree 0.

We first observe that $\mathrm{R}f_\star \mathscr{F}_i/\mathscr{F}_{i-1}(-D_{K_p})$ is a sheaf concentrated in degree 0 by an analogue of theorem 4.6 which only relies on the structure of the map f at the boundary.

Since $\mathcal{L}$ is very ample on $\mathfrak{S}^{\star,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}$ and $f_\star \mathscr{F}_i/\mathscr{F}_{i-1}(-D_{K_p})$ runs through a finite family of coherent sheaves, it follows that there exists m such that $\mathrm{R}\Gamma(\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}, \mathcal{L}^m \otimes \mathscr{F}_i/\mathscr{F}_{i-1}(-D_{K_p})) = 0$.

Let $L = \mathrm{H}^0(\mathfrak{S}^{\star,mod}_{K'_pK^p,\mathfrak{U}}, \mathcal{L}[1/p])$. This is a rank 1 projective $\mathscr{O}_{\mathcal{S}^\star_{K'_pK^p}}((\pi_{HT,K'_p})^{-1}(\mathcal{U}))$-module. By inverting p, we deduce that

$$\mathrm{R}\Gamma(\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}, \mathcal{L}^m \otimes \mathfrak{V}^{n-an,+}_{\nu_A}(-D_{K_p})[1/p]) = L^m \otimes_{\mathscr{O}_{\pi^{-1}_{HT,K'_p}(\mathcal{U})}} \mathrm{R}\Gamma(\pi^{-1}_{HT,K'_p}(\mathcal{U}), \mathfrak{V}^{n-an,+}_{\nu_A}(-D_{K_p})[1/p])$$

and therefore $\mathrm{R}\Gamma(\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}, \mathfrak{V}^{n-an,+}_{\nu_A}(-D_{K_p})[1/p])$ is concentrated in degree 0. Taking a Zariski covering of $\mathfrak{S}^{tor,mod}_{K'_pK^p,\Sigma,\mathfrak{U}}$, we deduce that the associated augmented Čech complex on the generic fiber is exact. By theorem 2.12, affinoids are acyclic for Banach sheaves arisings as generic fibers of flat small formal Banach sheaves.

We deduce that $\mathrm{R}\Gamma((\pi^{tor}_{HT,K'_p})^{-1}(\mathcal{U}), \mathcal{V}^{n-an,+}_{\nu_A}(-D_{K_p})) = 0$. The lemma follows by taking the invariants under K_p/K'_p. $\square$

Lemma 6.31. *Let $M_0 \stackrel{f_0}{\to} M_1 \stackrel{f_1}{\to} M_2$ be a complex of complete separated and torsion free A^+-modules. Assume that*

$$M_0 \otimes_{A^+} A^+/A^{++} \to M_1 \otimes_{A^+} A^+/A^{++} \to M_2 \otimes_{A^+} A^+/A^{++}$$

is exact. Then the complex $M_0 \stackrel{f_0}{\to} M_1 \stackrel{f_1}{\to} M_2$ is exact.

Proof. It follows from the assumptions that $\mathrm{Im}(f_0)+(A^{++}M_1 \cap \mathrm{Ker}(f_1)) = \mathrm{Ker}(f_1)$. Since M_2 is torsion free, we deduce that $(A^{++}M_1 \cap \mathrm{Ker}(f_1)) = A^{++}\mathrm{Ker}(f_1)$. Let $m \in \mathrm{Ker}(f_1)$. By successive approximation, we can construct a sequence of elements $m_n \in M_0$ for $n \geq 0$, with $m_n - m_{n+1} \in (A^{++})^n M_0$ and $f_0(m_n) - m \in (A^{++})^{n+1} M_1$. The sequence m_n converges to m_∞ and $f_0(m_\infty) = m$. $\square$

6.5. **The spectral sequence from locally analytic overconvergent to overconvergent cohomology.** Let $\nu = \nu_{alg}\chi : T(\mathbb{Z}_p) \to F^\times$ be a locally algebraic character where $\kappa_{alg} = -w_{0,M} w \nu_{alg} - (w_{0,M} w\rho + \rho)$ is M-dominant.

By propositions 6.18 and 6.19, we have morphisms:

$$\mathrm{R}\Gamma_w(K^p, \kappa_{alg}, \chi)^{+,fs} \to \mathrm{R}\Gamma_{w,an}(K^p, \nu)^{+,fs}$$

$$\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{-,fs} \to \mathrm{R}\Gamma_w(K^p, -w_{0,M}\kappa_{alg} - 2\rho_{nc}, \chi^{-1})^{-,fs}$$

and similarly for cuspidal cohomology. We remark that these morphisms are not Hecke equivariant. Namely, on $\mathrm{R}\Gamma_w(K^p, \kappa_{alg}, \chi)^{+,fs}$ we have an action of T^+ with the property that $T(\mathbb{Z}_p)$ acts via ν and on $\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{-,fs}$, we have an action of T^- such that $T(\mathbb{Z}_p)$ acts by $-\nu$ (see lemma 6.22). We deduce (compare with section 6.1.5) that there is a T^+-equivariant map : $\mathrm{R}\Gamma_w(K^p, \kappa_{alg}, \chi)^{+,fs} \to \mathrm{R}\Gamma_{w,an}(K^p, \nu)^{+,fs}(-\nu_{alg})$ and a T^--equivariant map : $\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{-,fs}(\nu_{alg}) \to \mathrm{R}\Gamma_w(K^p, -w_{0,M}\kappa_{alg} - 2\rho_{nc}, \chi^{-1})^{-,fs}$

We can study these maps with the help of the locally analytic BGG resolution.

Theorem 6.32. *In the setting above, there is a $\mathcal{H}^+_{p,m,0}$-equivariant spectral sequence $\mathbf{E}^{p,q}_w(K^p, \kappa, \chi)^+$ converging to finite slope overconvergent cohomology $\mathrm{H}^{p+q}_w(K^p, \kappa_{alg}, \chi)^{+,fs}$, such that*

$$\mathbf{E}^{p,q}_{w,1}(K^p, \kappa_{alg}, \chi)^+ =$$

$$\oplus_{v \in W_M, \ell(v)=p} \mathrm{H}^q_{w,an}\big(K^p, (((w_{0,M}w)^{-1} v w_{0,M} w)\cdot \nu_{alg})\chi\big)^{+,fs}\big(-(((w_{0,M}w)^{-1} v w_{0,M} w)\cdot \nu_{alg})\big).$$

There is a $\mathcal{H}^-_{p,m,0}$-equivariant spectral sequence $\mathbf{E}^{p,q}_w(K^p, \kappa, \chi)^-$ converging to finite slope overconvergent cohomology $\mathrm{H}^{p+q}_w(K^p, -w_{0,M}\kappa_{alg} - 2\rho_{nc}, \chi^{-1})^{-,fs}$, such that

$$\mathbf{E}^{p,q}_{w,1}(K^p, \kappa_{alg}, \chi)^- =$$

$$\oplus_{v \in W_M, \ell(v)=-p} \mathrm{H}^q_{w,an}\big(K^p, (((w_{0,M}w)^{-1} v w_{0,M} w)\cdot \nu_{alg})\chi\big)^{-,fs}\big((((w_{0,M}w)^{-1} v w_{0,M} w)\cdot \nu_{alg})\big).$$

Proof. This is the spectral sequence associated to the BGG sequence of theorem 6.10, as well as its variant for distribution sheaves. We use the following identities : $\nu_{alg} + \rho = -w^{-1}w_{0,M}(\kappa_{alg} + \rho)$ and $-w^{-1}w_{0,M}(v. \cdot \kappa_{alg} + \rho) - \rho = ((w_{0,M}w)^{-1} v w_{0,M} w) \cdot \nu_{alg}$. $\square$

6.6. **Slope estimates and control theorem.** We now formulate a conjecture regarding the slopes of the locally analytic overconvergent cohomologies. This is of course consistant with conjecture 5.29.

Conjecture 6.33. *Fix $w \in {}^M W$, and let $\nu : T(\mathbb{Z}_p) \to \mathbb{C}_p^\times$ be a continuous character.*

(1) *For any character λ of T^+ on $\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{+,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p, \nu, cusp)^{+,fs}$, we have $v(\lambda) \geq 0$.*
(2) *For any character λ of T^- on $\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{-,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p, \nu, cusp)^{-,fs}$, we have $v(\lambda) \leq 0$.*

Remark 6.34. The inequalities in conjecture 6.33 are compatible with those of conjecture 5.29 due to the way we have renormalized the Hecke operators acting on locally analytic cohomology. Similarly the slightly weaker bounds we prove in theorem 6.35 are compatible with those of theorem 5.33.

We can prove a bound which is slightly weaker than the conjecture.

Theorem 6.35. *Fix $w \in {}^M W$, and let $\nu : T(\mathbb{Z}_p) \to \mathbb{C}_p^\times$ be a continuous character.*

(1) *For any character λ of T^+ on $\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{+,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p, \nu, cusp)^{+,fs}$, we have $v(\lambda) \geq -w^{-1}w_{0,M}\rho - \rho$.*
(2) *For any character λ of T^- on $\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{-,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p, \nu, cusp)^{-,fs}$, we have $v(\lambda) \leq -w^{-1}\rho + \rho$.*

Proof. This is similar to the proof of theorem 5.33 and left to the reader. □

Corollary 6.36. *Fix $w \in {}^M W$ and let $\nu = \nu_{alg}\chi$ be a locally algebraic character. Let $\kappa_{alg} = -w_{0,M} w \nu_{alg} - (w_{0,M} w\rho + \rho)$ and suppose $\kappa_{alg} \in X^\star(T)^{M,+}$. Then the morphisms*

$$\mathrm{R}\Gamma_w(K^p, \kappa_{alg}, \chi)^{+, sss_{M,w}(\kappa_{alg})} \to \mathrm{R}\Gamma_{w,an}(K^p, \nu)^{+, nsss_{M,w}(\kappa_{alg})}$$

$$\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{-, nsss_M(-w_{0,M}\kappa_{alg} - 2\rho_{nc})} \to \mathrm{R}\Gamma_w(K^p, -w_{0,M}\kappa - 2\rho_{nc}, \chi^{-1})^{-, sss_M(-w_{0,M}\kappa_{alg} - 2\rho_{nc})}$$

and the corresponding morphisms for cuspidal cohomology are all quasi-isomorphisms.

Proof. This follows from the spectral sequence of theorem 6.32, together with the slope bounds of theorem 6.35. □

Remark 6.37. If we assume conjectures 5.29 and 6.33, we can replace the sss_M condition by the ss_M condition in the above corollary.

6.7. **Cup products.** We now consider cup-products on locally analytic overconvergent cohomology.

Theorem 6.38. *For all $w \in {}^M W$ and weights ν_A there is a pairing:*

$$\langle , \rangle : \mathrm{H}^i_{w,an}(K_p, \nu_A, cusp)^{\pm, fs} \times \mathrm{H}^{d-i}_{w,an}(K_p, \nu_A)^{\mp, fs} \to A$$

Let $\nu = \nu_{alg}\chi$ be a locally algebraic weight so that $\kappa_{alg} = -w_{0,M} w \nu_{alg} - (w_{0,M} w\rho + \rho)$ is M-dominant. The above pairing induces a pairing between the spectral sequences:

$$\langle , \rangle_{p,q,r} : \mathbf{E}^{p,q}_{w,r}(K^p, \kappa_{alg}, \chi, cusp)^{\pm} \times \mathbf{E}^{-p,d-q}_{w,r}(K^p, \kappa, \chi^{-1})^{\mp} \to F$$

On the abutment of the spectral sequence the pairing $\langle , \rangle_{p,q,\infty}$ is induced by the pairing of theorem 5.22:

$$\mathrm{H}^{p+q}_w(K^p, \kappa, \chi, cusp)^{\pm, fs} \times \mathrm{H}^{d-p-q}_w(K^p, -2\rho_{nc} - w_{0,M}\kappa, \chi^{-1})^{\mp, fs} \to F.$$

Proof. We construct the pairing

$$\langle,\rangle : \mathrm{H}^i_{w,an}(K_p,\nu_A,cusp)^{+,fs}\times \mathrm{H}^{d-i}_{w,an}(K_p,\nu_A)^{-,fs}\to F.$$

We can realize $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{+,fs}$ as the the finite slope part of

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{s,\overline{s}}K_{p,m',0})}((\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{s,-1}K_{p,m',0}),\mathcal{V}^{n-an}_{\nu_A}(-D))$$

for $s >> 0$ and $m' >> s$ by definition 6.25 and theorem 6.26. Similarly we can realize $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A)^{-,fs}$ as the finite slope part of

$$\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{\overline{s+1},s-1}K_{p,m',0})}((\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{-1,s-1}K_{p,m',0}),\mathcal{D}^{n-an}_{\nu_A}).$$

Moreover, by construction we have a pairing $\mathcal{V}^{n-an}_{\nu_A}(-D)\times\mathcal{D}^{n-an}_{\nu_A}\to\mathcal{V}_{-2\rho_{nc}}(-D)\otimes A$. We have a cup-product by proposition 2.3:

$$\mathrm{H}^i_{(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{s,\overline{s}}K_{p,m',0})}((\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{s,-1}K_{p,m',0}),\mathcal{V}^{n-an}_{\nu_A}(-D))\times$$

$$\mathrm{H}^{d-i}_{(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{\overline{s+1},s-1}K_{p,m',0})}((\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{-1,s-1}K_{p,m',0}),\mathcal{D}^{n-an}_{\nu_A})$$

$$\to \mathrm{H}^{d}_{(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{\overline{s+1},\overline{s}}K_{p,m',0})}((\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{s,s-1}K_{p,m',0},\mathcal{V}_{-2\rho_{nc}}(-D)\otimes A)$$

and there is a trace map (by theorem 2.32):

$$\mathrm{H}^{d}_{(\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{\overline{s+1},\overline{s}}K_{p,m',0})}((\pi^{tor}_{HT,K_{p,m',0}})^{-1}(]C_{w,k}[_{s,s-1}K_{p,m',0}),\mathcal{V}_{-2\rho_{nc}}(-D)\otimes A)\to A.$$

This pairing intertwines the actions of $\mathcal{H}^+_{p,m,0}$ and $\mathcal{H}^-_{p,m,0}$. It is straightforward (and painful) to check that the induced pairing

$$\langle,\rangle : \mathrm{H}^i_{w,an}(K_p,\nu_A,cusp)^{+,fs}\times \mathrm{H}^{d-i}_{w,an}(K_p,\nu_A)^{-,fs}\to A$$

is independent of choices.

The rest of the theorem follows from the functoriality of the trace map.

□

6.8. **Eigenvarieties.** Consider the Iwasawa algebra $\mathbb{Z}_p[[T^c(\mathbb{Z}_p)]]$ and the weight space

$$\mathcal{W}=\mathrm{Spa}(\mathbb{Z}_p[[T^c(\mathbb{Z}_p)]],\mathbb{Z}_p[[T^c(\mathbb{Z}_p)]])\times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)}\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p).$$

For an adic space $\mathrm{Spa}(A,A^+)$, we have:

$$\mathrm{Hom}(\mathrm{Spa}(A,A^+),\mathcal{W})=\{\text{Continuous characters } \nu_A : T^c(\mathbb{Z}_p)\to A^\times\}.$$

Let also $\widehat{T}$ be the analytic adic space of characters of $T(\mathbb{Q}_p)$, whose restriction to $T(\mathbb{Z}_p)$ factor through $T^c(\mathbb{Z}_p)$. If we fix a splitting ξ for the map $T(\mathbb{Q}_p)\to T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$, and we fix an isomorphism $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)\simeq\mathbb{Z}^r$ (for a suitable integer r), then we have an isomorphism $\widehat{T}\simeq\mathcal{W}\times(\mathbb{G}^{an}_m)^r$, where the map sends a character λ to $(\lambda|_{T^c(\mathbb{Z}_p)},\lambda(\xi(e_1)),\cdots,\lambda(\xi(e_r)))$ (for the canonical basis $e_1,\cdots,e_r$ of $\mathbb{Z}^r$). We also observe that there is a natural map $T\to\mathscr{O}_{\widehat{T}}$.

Let $\mathrm{Spa}(A,A^+)\subset\mathcal{W}$ be an affinoid open and let $\nu^{un}_A : T^c(\mathbb{Z}_p)\to A^\times$ be the universal character.

For each $w\in {}^MW$ we have four objects of $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{proj}(A))$: $\mathrm{R}\Gamma_{w,an}(K^p,\nu^{un}_A)^{\pm,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p,\nu^{un}_A,cusp)^{\pm,fs}$. We define an action of T^+ on all these spaces. For $\mathrm{R}\Gamma_{w,an}(K^p,\nu^{un}_A)^{+,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p,\nu^{un}_A,cusp)^{+,fs}$ we use the existing action (defined using Hecke operators). For $\mathrm{R}\Gamma_{w,an}(K^p,\nu^{un}_A)^{-,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p,\nu^{un}_A,cusp)^{-,fs}$ we compose the action of T^- (defined using Hecke operators) which we compose with the isomorphism $T^+\to T^-$ given by $t\mapsto t^{-1}$. We observe that the action of T^+ extends to a $T(\mathbb{Q}_p)$-action, and that its restriction to $T(\mathbb{Z}_p)$ is given by the character ν_A^{un}.

In what follows we will denote these cohomologies by $\mathcal{M}_w^{\bullet,\pm,fs}$ and $\mathcal{M}_{w,cusp}^{\bullet,\pm,fs}$ in order to shorten notation.

Remark 6.39. Our weight variable ν is morally a translate of the infinitesimal character. We recall briefly how to switch between the weight of our coherent cohomologies and the infinitesimal character. Fix $\nu = \nu^{alg}\chi$ be a locally algebraic weight. Then $\mathcal{M}_w^{\bullet,+,fs}|_\nu$ and $\mathcal{M}_{w,cusp}^{\bullet,+,fs}|_\nu$ are related to classical or overconvergent cohomology in weight κ_{alg} where $\nu_{alg} = -w^{-1}w_{0,M}(\kappa_{alg}+\rho)-\rho$. On the other hand, $\mathcal{M}_w^{\bullet,-,fs}|_\nu$ and $\mathcal{M}_{w,cusp}^{\bullet,-,fs}|_\nu$ are related to classical or overconvergent cohomology in weight $\kappa_{alg}^\vee = -w_{0,M}\kappa_{alg} - 2\rho_{nc}$ and $\nu_{alg} = w^{-1}(\kappa_{alg}^\vee + \rho) - \rho$.

We fix $t \in T^{++}$ and choose compact lifts $\tilde{T}$ of $t^{\pm 1}$ acting on $\mathcal{M}_w^{\bullet,\pm,fs}$ and $\mathcal{M}_{w,cusp}^{\bullet,\pm,fs}$. By the relative spectral theory (sections 6.3.1, 6.3.2, 6.3.3), there is a (non-canonical) spectral variety $\pi : \tilde{\mathcal{Z}}_t \to \mathrm{Spa}(A, A^+)$, over which, there are for each $w \in {}^MW$ four complexes, perfect as complexes of $\pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)}$-modules, $\tilde{\mathcal{M}}_{w,t}^{\bullet,\pm,fs}$, $\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,\pm,fs}$ with the property that $\pi_\star\tilde{\mathcal{M}}_{w,t}^{\bullet,\pm,fs} = \mathcal{M}_w^{\bullet,\pm,fs}$, $\pi_\star\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,\pm,fs} = \mathcal{M}_{w,cusp}^{\bullet,\pm,fs}$.

We also have morphisms

$$\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,\pm,fs} \to \tilde{\mathcal{M}}_{w,t}^{\bullet,\pm,fs}$$

as well as cup-products given by theorem 6.38:

$$\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,\pm,fs}[d] \otimes^L_{\pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)}} \tilde{\mathcal{M}}_{w,t}^{\bullet,\mp,fs} \to \pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)}.$$

Let us also define

$$\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,+,fs,\vee} = \mathrm{RHom}_{\pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)}}(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,+,fs}, \pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)})[-d].$$

Then we have morphisms of $\mathscr{O}_{\tilde{\mathcal{Z}}_t}$-modules

$$\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,-,fs} \to \tilde{\mathcal{M}}_{w,t}^{\bullet,-,fs} \to \tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,+,fs,\vee}$$

where the second morphism is induced by the cup product.

Passing to cohomology, we get graded coherent sheaves over $\tilde{\mathcal{Z}}_t$: $\oplus_k \mathrm{H}^k(\tilde{\mathcal{M}}_{w,t}^{\bullet,\pm,fs})$, $\oplus_k \mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,\pm,fs})$, and $\oplus_k \mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,+,fs,\vee})$.

We have maps

$$\mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,\pm,fs}) \to \mathrm{H}^k(\tilde{\mathcal{M}}_{w,t}^{\bullet,\pm,fs})$$

as well as pairings

$$\mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,\pm,fs}) \otimes \mathrm{H}^{d-k}(\tilde{\mathcal{M}}_{w,t}^{\bullet,\mp,fs}) \to \pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)}$$

and there is also a map

$$\mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,-,fs}) \to \mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,+,fs,\vee})$$

We let $\mathcal{Z}_t$ be the closed subspace of $\tilde{\mathcal{Z}}_{w,t}$ equal to the support of these sheaves. We let $\mathcal{Z}$ be the finite $\mathcal{Z}_t$-adic space whose algebra is the coherent $\mathscr{O}_{\mathcal{Z}_{w,t}}$-algebra generated by the operators $t' \in T^+$ acting on

$$\bigoplus_{w \in {}^MW, k \in \mathbb{Z}} \Big(\mathrm{H}^k(\tilde{\mathcal{M}}_{w,t}^{\bullet,+,fs}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}_{w,t}^{\bullet,-,fs}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,+,fs}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}_{w,cusp,t}^{\bullet,-,fs})\Big)$$

where in the case of the $-$ sheaves we let $t \in T^+$ act by t^{-1}.

This is the spectral variety. We now assume that $K^p = \prod K_\ell$ and we let S' be the set of primes $\ell \neq p$ such that K_ℓ is not hyperspecial. We let $S = S' \cup \{p\}$. Let $\mathcal{H}^S = \mathcal{C}_c^\infty(G(\mathbb{A}_f^S)//K^S, \mathbb{Q})$ be the spherical Hecke algebra at places away from S. We let $\mathcal{E}$ be the finite $\mathcal{Z}$-adic space whose algebra is the coherent $\mathscr{O}_{\mathcal{Z}_t}$-algebra generated by the operators $t \in T^+$ and $h \in \mathcal{H}^S$ acting on

$$\bigoplus_{w \in {}^MW, k \in \mathbb{Z}} \Big(\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,+,fs}_{w,t}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,-,fs}_{w,t}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,+,fs}_{w,cusp,t}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,-,fs}_{w,cusp,t}) \Big)$$

where in the case of the $-$ sheaves we let $t \in T^+$ and $h \in \mathcal{H}^S$ act by t^{-1} and h^t.

We denote by $\oplus_k \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,\pm,fs}_{w})$ and $\oplus_k \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,\pm,fs}_{w,cusp})$ the coherent sheaves on $\mathcal{E}$ whose pushforwards to $\mathcal{Z}_t$ are $\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,\pm,fs}_{w,t})$ and $\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,\pm,fs}_{w,cusp,t})$. They do not depend on the choice of t and $\tilde{T}$.

The spaces $\mathcal{Z}$, $\mathcal{E}$, and the graded sheaves on them are defined over $\mathrm{Spa}(A, A^+)$. By letting $\mathrm{Spa}(A, A^+)$ run through a cover of $\mathcal{W}$ we can glue them. We now change our notation and consider $\mathcal{Z}$ and $\mathcal{E}$, together with their graded sheaves over $\mathcal{W}$. Note that $\mathcal{Z}$ is by construction a closed subspace of $\widehat{T}$.

A (classical rigid analytic) point of $\mathcal{Z}$ corresponds to a character $\lambda_p : T(\mathbb{Q}_p) \to \bar{F}^\times$. We can attach to λ_p the weight $\nu = \lambda_p|_{T^c(\mathbb{Z}_p)} : T^c(\mathbb{Z}_p) \to \bar{F}^\times$. When the weight $\nu = \nu_{alg}\chi$ is locally algebraic, then we let $\lambda_p^{sm} = \lambda_p \nu_{alg}^{-1}$ with ν_{alg} viewed as a character of $T(\mathbb{Q}_p)$. Then λ_p^{sm} factors through a character of $T(\mathbb{Q}_p)/T_b(\mathbb{Z}_p)$ where b is the conductor of χ.

Remark 6.40. The supscript sm stands for smooth, because the character λ_p^{sm} is the smooth character attached to λ_p. Remark that the Hecke action on classical cohomology produces smooth characters. This is visible in point (1) of theorem 6.41 below.

A point of $\mathcal{E}$ corresponds to a pair (λ_p, λ^S) where $\lambda^S : \mathcal{H}^S \to \bar{F}$ is a character.

Theorem 6.41. *The eigenvariety $\pi : \mathcal{E} \to \mathcal{W}$ is locally quasi-finite and partially proper. It carries graded coherent sheaves*

$$\bigoplus_{w \in {}^MW, k \in \mathbb{Z}} \Big(\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,+,fs}_{w}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,-,fs}_{w}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,+,fs}_{w,cusp}) \oplus \mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,-,fs}_{w,cusp}) \Big)$$

and they satisfy the following properties:

(1) *(Any classical, finite slope eigenclass gives a point of the eigenvariety) For any $\kappa_{alg} \in X^\star(T^c)^{M,+}$, finite order character $\chi : T^c(\mathbb{Z}_p) \to \overline{F}^\times$, and any system of Hecke eigenvalues $(\lambda_p^{sm}, \lambda^S)$ occurring in $\mathrm{H}^i(K^p, \kappa_{alg}, \chi)^{+,fs}$ (resp. $\mathrm{H}^i(K^p, \kappa_{alg}^\vee, \chi^{-1})^{-,fs}$, $\mathrm{H}^i(K^p, \kappa_{alg}, \chi, cusp)^{+,fs}$, or $\mathrm{H}^i(K^p, \kappa^\vee_{alg}, \chi^{-1}, cusp)^{-,fs}$) there is a $w = w_M w^M \in W$, so that if $\nu = \nu_{alg}\chi$ with $\nu_{alg} = -w^{-1}w_{0,M}(\kappa_{alg} + \rho) - \rho$, then $(\lambda_p^{class}\nu_{alg}, \lambda^S)$ is a point of the eigenvariety $\mathcal{E}$ which lies in the support of $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,+,fs}_{w^M})$ (resp. $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,-,fs}_{w^M})$, $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,+,fs}_{w^M,cusp})$, or $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,-,fs}_{w^M,cusp})$).*

(2) *(Small slope points of the eigenvariety in regular, locally algebraic weights are classical) Conversely if $\nu = \nu_{alg}\chi$ is a locally algebraic weight with $\nu_{alg} \in X^\star(T)^+$, and (λ_p, λ^S) is a point of $\mathcal{E}$ in the support of $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,+,fs}_{w})$ (resp. $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,-,fs}_{w})$, $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,+,fs}_{w,cusp})$, or $\oplus_{k\in\mathbb{Z}} H^k(\tilde{M}^{\bullet,-,fs}_{w,cusp})$) for some $w \in {}^MW$, and if λ_p satisfies $+, sss_w(\nu)$ then $(\lambda_p^{sm}, \lambda^S)$ occurs in*

$\mathrm{H}^i(K^p,\kappa_{alg},\chi)^{+,fs}$ (resp. $\mathrm{H}^i(K^p,\kappa^\vee_{alg},\chi^{-1})^{-,fs}$, $\mathrm{H}^i(K^p,\kappa_{alg},\chi,cusp)^{+,fs}$, or $\mathrm{H}^i(K^p,\kappa^\vee_{alg},\chi^{-1},cusp)^{-,fs}$) for $\kappa_{alg}=-w_{0,M}w(\nu_{alg}+\rho)-\rho$.

(3) *(Serre duality interpolates over the eigenvariety) We have pairings:*

$$\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,\pm,fs}_{w,cusp})\otimes\mathrm{H}^{d-k}(\tilde{\mathcal{M}}^{\bullet,\mp,fs}_{w})\to\pi^{-1}\mathscr{O}_{\mathcal{W}}.$$

and these pairings are compatible with Serre duality under the classicality theorem.

Proof. We only prove the first point for $\mathrm{H}^i(K^p,\kappa_{alg},\chi)^{+,fs}$. The other cases are similar. There is a succession of three spectral sequences, going from the sheaves $\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,+,fs}_w)$ (for varying k and w) to the classical cohomology $\mathrm{H}^i(K^p,\kappa_{alg},\chi)^{+,fs}$. The first spectral sequence is a tor spectral sequence (see [Han17], thm. 3.3.1 for example):

$$E_2^{p,q}=\mathrm{Tor}_{-p}^{\pi^{-1}(\mathscr{O}_{\mathcal{W}})}(\mathrm{H}^p(\tilde{\mathcal{M}}^{\bullet,+,fs}_w),k(\nu))\Rightarrow\mathrm{H}^{p+q}_{w,an}(K^p,\nu)^{+,fs}$$

The second spectral sequence is the spectral sequence of theorem 6.32, from locally analytic overconvergent cohomology to overconvergent cohomology. The third spectral sequence is the spectral sequence of theorem 5.15, from overconvergent to classical cohomology. Therefore, starting from a classical class, we can lift is successively to the E_1 terms of the last two spectral sequences (for suitable choices of w_M and w^M) and then to a class in a suitable $\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,+,fs}_w)_\nu$. We now prove point (2). Let $(\nu,\lambda_p,\lambda^S)$ be a point in the support of $\oplus_{k\in\mathbb{Z}}H^k(\tilde{M}^{\bullet,+,fs}_w)$ over ν. Then on sees that there is a corresponding class in $\oplus_k\mathrm{H}^k_{w,an}(K^p,\nu)^{+,fs}$. Indeed, if there were no such class, a version of Nakayama's lemma for perfect complexes would contradict the existence of $(\nu,\lambda_p,\lambda^S)$ in the support of $\oplus_{k\in\mathbb{Z}}H^k(\tilde{M}^{\bullet,+,fs}_w)$. Then we conclude by the control theorems (corollary 6.36 and theorem 5.66):

$$\oplus_k\mathrm{H}^k_{w,an}(K^p,\nu)^{+,sss_w(\nu)}=\oplus_k\mathrm{H}^k(K^p,-w_{0,M}w(\nu_{alg}+\rho)-\rho,\chi)^{+,sss_w(\nu)}.$$

The last point follows from the functoriality of the pairing. ☐

We now define certain components of the eigenvariety of maximal dimension and show that they contain all finite slope interior cohomology classes.

By theorem 6.29, $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{\pm,fs}$ is concentrated in degree $[0,\ell_\pm(w)]$. This implies that $\oplus_k\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,+,fs}_{w,cusp})$ is concentrated in degrees $[0,\ell_\pm(w)]$ and that $\oplus_k\mathrm{H}^k(\tilde{\mathcal{M}}^{\bullet,\pm,fs,\vee}_{w,cusp})$ is concentrated in degree $[\ell_\pm(w),d]$. Let us define the following coherent sheaf over $\mathcal{E}$:

$$\overline{\mathcal{C}ous}_w(K^p)^\pm=\mathrm{Im}(\mathrm{H}^{\ell_\pm(w)}(\tilde{\mathcal{M}}^{\bullet,-,fs}_{w,cusp})\to\mathrm{H}^{\ell_-(w)}(\tilde{\mathcal{M}}^{\bullet,\pm,fs,\vee}_{w,cusp})).$$

We have pairings :

$$\overline{\mathcal{C}ous}_w(K^p)^+\times\overline{\mathcal{C}ous}_w(K^p)^-\to\pi^{-1}\mathscr{O}_{\mathcal{W}}$$

Let $\mathcal{Z}^!$ (resp. $\mathcal{E}^!$) be the closed subspace of $\mathcal{Z}$ (resp $\mathcal{E}$) equal to the support of $\bigoplus_{w\in{}^MW}\overline{\mathcal{C}ous}_w(K^p)^\pm$. For $w\in{}^MW$ we also write $\mathcal{Z}^!_w$ (resp. $\mathcal{E}^!_w$) for the supports of the individual sheaf $\overline{\mathcal{C}ous}_w(K^p)^\pm$.

Recall also, for any $\kappa_{alg}\in X^\star(T^c)^{M,+}$, $\chi:T^c(\mathbb{Z}_p)\to F^\times$ a finite order character, that we have the interior Cousin complex : $\overline{\mathcal{C}ous}(K^p,\kappa_{alg},\chi)^\pm$ where the module placed in degree i is

$$\oplus_{w\in{}^MW,\ell_\pm(w)=i}\mathrm{Im}(\mathrm{H}^{\ell_\pm(w)}_w(K_p,\kappa,\chi,cusp)^{+,fs}\to\mathrm{H}^{\ell_\mp(w)}_w(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{+,fs,\vee}).$$

and recall that we defined

$$\overline{\mathrm{H}}^{\ell_{\pm}(w)}_{w}(K^p,\kappa_{alg},\chi)^{\pm,fs} = \mathrm{Im}(\mathrm{H}^{\ell_{\pm}(w)}_{w}(K_p,\kappa,\chi,cusp)^{\pm,fs} \to \mathrm{H}^{\ell_{\mp}(w)}_{w}(K_p,-2\rho_{nc}-w_{0,M}\kappa,\chi^{-1},cusp)^{\mp,fs,\vee}).$$

We aso have to consider the analytic version. We let

$$\overline{\mathrm{H}}^{\ell_{\pm}(w)}_{w,an}(K^p,\nu)^{\pm,fs} = \mathrm{Im}(\mathrm{H}^{\ell_{\pm}(w)}_{w,an}(K_p,\nu,cusp)^{\pm,fs} \to \mathrm{H}^{\ell_{\mp}(w)}_{w,an}(K_p,\nu,cusp)^{\mp,fs,\vee}).$$

Theorem 6.42. *The following holds:*

(1) *The coherent sheaves $\overline{\mathcal{C}ous}_w(K^p)^{\pm}$ are $\pi^{-1}\mathscr{O}_{\mathcal{W}}$-torsion free.*

(2) *The eigenvariety $\mathcal{E}^!$ and the spectral variety $\mathcal{Z}^!$ are either empty or equidimensional of dimension $\dim \mathcal{W}$. The same is true of the $\mathcal{E}^!_w$ and $\mathcal{Z}^!_w$ for each $w \in {}^MW$.*

(3) *For all $w \in {}^MW$ and $\kappa_{alg} \in X^\star(T^c)^{M,+}$, $\chi : T^c(\mathbb{Z}_p) \to F^\times$ a finite order character, let $\nu_{alg} = -w^{-1}w_{0,M}(\kappa_{alg}+\rho)-\rho$, there are canonical surjective maps*

$$\overline{\mathcal{C}ous}^{-}_{w}|_{\nu} \to \overline{\mathrm{H}}^{\ell_-(w)}_{w,an}(K^p,\nu)^{-,fs} \to \overline{\mathrm{H}}^{\ell_-(w)}_{w}(K^p,\kappa^\vee_{alg},\chi^{-1})^{-,fs}.$$

In particular, any eigenclass in $\overline{\mathrm{H}}^{\ell_-(w)}_{w}(K^p,\kappa^\vee_{alg},\chi^{-1})^{-,fs}$ gives a point on $\mathcal{E}^!_w$ of weight $\nu = \nu_{alg}\chi$.

(4) *There is also a canonical surjective map*

$$\overline{\mathcal{C}ous}_w(K^p)^+|_{\nu} \to \overline{\mathrm{H}}^{\ell(w)}_{w,an}(K^p,\nu)^{+,fs}$$

and a canonical map:

$$\overline{\mathrm{H}}^{\ell(w)}_{w}(K^p,\kappa_{alg},\chi)^{+,fs} \to \overline{\mathrm{H}}^{\ell(w)}_{w,an}(K^p,\nu)^{+,fs}$$

inducing an isomorphism :

$$\overline{\mathrm{H}}^{\ell(w)}_{w}(K^p,\kappa_{alg},\chi)^{+,sss_{M,w}(\kappa^{alg}),sss_{M,w}(\kappa^{alg})^\vee} \to \overline{\mathrm{H}}^{\ell(w)}_{w,an}(K^p,\nu)^{+,sss_{M,w}(\kappa^{alg}),sss_{M,w}(\kappa^{alg})^\vee}.$$

and there is therefore a surjection :

$$\overline{\mathcal{C}ous}_w(K^p)^+|_{\nu}^{+,sss_{M,w}(\kappa^{alg}),sss_{M,w}(\kappa^{alg})^\vee} \to \overline{\mathrm{H}}^{\ell(w)}_{w}(K^p,\kappa_{alg},\chi)^{+,sss_{M,w}(\kappa^{alg}),sss_{M,w}(\kappa^{alg})^\vee}.$$

(5) *The pairing $\overline{\mathcal{C}ous}_w(K^p)^+ \times \overline{\mathcal{C}ous}_w(K^p)^- \to \pi^{-1}\mathscr{O}_{\mathcal{W}}$ is compatible with the pairing $\overline{\mathrm{H}}^{\ell(w)}_{w,an}(K^p,\nu)^{+,fs} \times \overline{\mathrm{H}}^{\ell_-(w)}_{w,an}(K^p,\nu)^{-,fs} \to F^\times$ via the specialization maps*

$$\overline{\mathcal{C}ous}_w(K^p)^+|_{\nu} \to \overline{\mathrm{H}}^{\ell(w)}_{w,an}(K^p,\nu)^{+,fs}$$

and

$$\overline{\mathcal{C}ous}_w(K^p)^-|_{\nu} \to \overline{\mathrm{H}}^{\ell_-(w)}_{w,an}(K^p,\nu)^{-,fs},$$

and compatible with the pairing $\overline{\mathrm{H}}^{\ell(w)}_{w}(K^p,\kappa_{alg},\chi)^{+,fs} \times \overline{\mathrm{H}}^{\ell_-(w)}_{w}(K^p,\kappa^\vee_{alg},\chi^{-1})^{-,fs} \to F^\times$ via the maps :

$$\overline{\mathrm{H}}^{\ell(w)}_{w}(K^p,\kappa_{alg},\chi)^{+,fs} \to \overline{\mathrm{H}}^{\ell(w)}_{w,an}(K^p,\nu)^{+,fs}$$

and

$$\overline{\mathrm{H}}^{\ell_-(w)}_{w,an}(K^p,\nu)^{-,fs} \to \overline{\mathrm{H}}^{\ell_-(w)}_{w}(K^p,\kappa^\vee_{alg},\chi^{-1})^{-,fs}.$$

Proof. Since $\mathrm{H}^{\ell_\pm(w)}(\tilde{\mathcal{M}}^{\bullet, fs,\vee}_{w,cusp})$ is the first non-zero cohomology group (assuming it is indeed non-zero), it is torsion free as a $\pi^{-1}\mathscr{O}_{\mathcal{W}}$-module. We deduce that

$$\mathrm{Im}(\mathrm{H}^{\ell_\pm(w)}(\tilde{\mathcal{M}}^{\bullet,\pm,fs}_{w,cusp}) \to \mathrm{H}^{\ell_\pm(w)}(\tilde{\mathcal{M}}^{\bullet,\pm,fs,\vee}_{w,cusp})) = \overline{\mathcal{C}ous}_w(K^p)^{\pm}$$

also has this property. The first and second point follows. For the third point we first fix an open $\mathrm{Spa}(A, A^+)$ of $\mathcal{W}$ which contains ν. We have a commutative diagram where the bottom vertical map are specialization in weight ν and the top vertical map are specialization from overconvergent analytic to overconvergent cohomology, while the horizontal maps are induced from the pairing:

$$\begin{array}{ccc}
\mathrm{H}^{\ell_-(w)}_w(K_p, -w_{0,M}\kappa_{alg} - 2\rho_{nc}, \chi^{-1}, cusp)^{-,fs} & \longrightarrow & (\mathrm{H}^{\ell_+(w)}_w(K_p, \kappa_{alg}, \chi, cusp)^{+,fs})^\vee \\
\uparrow & & \uparrow \\
\mathrm{H}^{\ell_-(w)}_{w,an}(K_p, \nu, cusp)^{-,fs} & \longrightarrow & (\mathrm{H}^{\ell_+(w)}_{w,an}(K_p, \nu, cusp)^{+,fs})^\vee \\
\uparrow & & \uparrow \\
\mathrm{H}^{\ell_-(w)}_{w,an}(K_p, \nu^{un}_A, cusp)^{-,fs} & \longrightarrow & \mathrm{Ext}^{-\ell_+(w)}(\mathrm{R}\Gamma_{w,an}(K_p, \nu^{un}_A, cusp)^{+,fs}), A)
\end{array}$$

The left vertical maps are surjective because they are induced by taking cohomology of surjective maps of sheaves, and the cohomology above degree $l_-(w)$ vanishes. We therefore deduce that the class c can be lifted in an Hecke-equivariant way to a section of

$$\mathrm{Im}(\mathrm{H}^{\ell_-(w)}(\tilde{\mathcal{M}}^{\bullet,-,fs}_{w,cusp}) \to \mathrm{H}^{\ell_-(w)}(\tilde{\mathcal{M}}^{\bullet,+,fs,\vee}_{w,cusp}))|_\nu.$$

The proof of the fourth point is very similar, the diagram is now :

$$\begin{array}{ccc}
\mathrm{H}^{\ell_+(w)}_w(K_p, \kappa_{alg}, \chi, cusp)^{+,fs} & \longrightarrow & (\mathrm{H}^{\ell_-(w)}_w(K_p, -w_{0,M}\kappa_{alg} - 2\rho_{nc}, \chi^{-1}, cusp)^{-,fs})^\vee \\
\downarrow & & \downarrow \\
\mathrm{H}^{\ell_+(w)}_{w,an}(K_p, \nu, cusp)^{+,fs} & \longrightarrow & (\mathrm{H}^{\ell_-(w)}_{w,an}(K_p, \nu, cusp)^{-,fs})^\vee \\
\uparrow & & \uparrow \\
\mathrm{H}^{\ell_+(w)}_{w,an}(K_p, \nu^{un}_A, cusp)^{+,fs} & \longrightarrow & \mathrm{Ext}^{-\ell_-(w)}(\mathrm{R}\Gamma_{w,an}(K_p, \nu^{un}_A, cusp)^{-,fs}), A)
\end{array}$$

and the condition $+, sss_{M,w}(\kappa^{alg}), sss_{M,w}(\kappa^{alg})^\vee$ turns the maps

$$\mathrm{H}^{\ell_+(w)}_w(K_p, \kappa_{alg}, \chi, cusp)^{+,fs} \to \mathrm{H}^{\ell_+(w)}_{w,an}(K_p, \nu, cusp)^{+,fs}$$

and

$$\mathrm{H}^{\ell_-(w)}_w(K_p, -w_{0,M}\kappa_{alg} - 2\rho_{nc}, \chi^{-1}, cusp)^{-,fs})^\vee \to \mathrm{H}^{\ell_-(w)}_{w,an}(K_p, \nu, cusp)^{-,fs})^\vee$$

into isomorphisms on the corresponding strongly small slope part. The last point follows from the functoriality of the pairing. $\square$

Remark 6.43. There is an asymmetry between point (3) and (4) which concern respectively the $-$ and $+$ theories. This asymmetry follows from our choice to develop the $+$ theory using locally analytic induction sheaves and the $-$ theory using their dual distribution sheaves.

Corollary 6.44. *Let $\overline{\mathcal{C}ous}(K^p, \kappa_{alg}^\vee, \chi^{-1})^-$ be the interior Cousin complex of section 5.8. Then for any eigenclass $c \in \overline{\mathcal{C}ous}(K^p, \kappa_{alg}^\vee, \chi^{-1})^-$ in degree i, there is a $w \in {}^MW$ with $\ell_-(w) = i$ so that c gives a point of $\mathcal{E}_w^!$ in weight $\nu = \nu_{alg}\chi$ where $\nu_{alg} = -w^{-1}w_{0,M}(\kappa_{alg} + \rho) - \rho$.*

Proof. This is immediate from theorem 6.42, given the definition of the degree i term of the Cousin complex given in section 5.8. □

We can now state the following theorem which asserts that any interior finite slope eigenclass admits an analytic deformation over the total weight space.

Theorem 6.45. *Let $\kappa_{alg} \in X^\star(T)^{M,+}$ and let $\chi : T(\mathbb{Z}_p) \to F^\times$ be a finite character. For any eigenclass in interior cohomology $c \in \overline{\mathrm{H}}^i(K_p, \kappa_{alg}^\vee, \chi^{-1})^{-,fs} = (\overline{\mathrm{H}}^{d-i}(K_p, \kappa, \chi)^{+,fs})^\vee$ there is a $w \in {}^MW$ with $\ell_-(w) = i$ so that c gives a point of $\mathcal{E}_w^!$ in weight $\nu = \nu_{alg}\chi$ where $\nu_{alg} = -w^{-1}w_{0,M}(\kappa_{alg} + \rho) - \rho$.*

Proof. By corollary 5.27, $\overline{\mathrm{H}}^i(K_p, -w_{0,M}\kappa - 2\rho_{nc}, \chi^{-1})^{-,fs}$ is a subquotient of $\mathrm{H}^i(\overline{\mathcal{C}ous}(K^p, -w_{0,M}\kappa - 2\rho_{nc}, \chi^-)^-)$, so the theorem follows from corollary 6.44.

□

To any such interior eigenclass c as in theorem 6.45, we can attach a subset ${}^MW(c)$ of ${}^MW$ which consists of all $w \in {}^MW$ such that c lifts to an eigenclass of $\overline{\mathcal{C}ous}_w(K^p)^-$ and therefore gives a point on $\mathcal{E}_w^!$. If $w \in {}^MW(c) \cap C(\kappa)^+$, then $\nu \in X^\star(T^c)^+$. It seems natural to ask for some condition under which ${}^MW(c) \cap C(\kappa)^+ \neq \emptyset$ or even for ${}^MW(c) = C(\kappa)^+$.

Proposition 6.46. *Assume that $\kappa + \rho$ is G-regular and let*

$$c \in \overline{\mathrm{H}}^i(K_p, \kappa^\vee, \chi^{-1})^{-,sss_M(\kappa),sss_M(\kappa)^\vee}.$$

Then ${}^MW(c) \subseteq C(\kappa)^+$. In particular, if $\kappa + \rho$ is G-regular, ${}^MW(c) = C(\kappa)^+$.

Proof. By corollary 5.68, we have that $\overline{\mathcal{C}ous}(K^p, \kappa^\vee, \chi^{-1})^{-,sss^M(\kappa),sss^M(\kappa)^\vee}$ is concentrated in the range $[\ell_{\min}(\kappa^\vee), \ell_{\max}(\kappa^\vee)]$ and the only objects appearing in the complex are simply the modules $\overline{\mathrm{H}}_w^{\ell_-(w)}(K^p, \kappa^\vee, \chi^{-1})^{-,sss^M(\kappa),sss^M(\kappa)^\vee}$ (in degree $\ell_-(w)$ for $w = C(\kappa)^+ = C(\kappa^\vee)^-$. Therefore the class c can only lift to a class in $\overline{\mathcal{C}ous}_w(K^p)^-$ for $w \in C(\kappa)^+$. The last statement follows from the property that if $\kappa + \rho$ is G-regular, $C(\kappa)^+$ contains only one element and ${}^MW(c)$ is always non empty.

□

Remark 6.47. If c is a cohomology class represented by an automorphic form π which is tempered at ∞ and contributes to coherent cohomology in weight $\kappa^\vee$, then π_∞ is a limit of discrete series which is described by the pair consisting of its infinitesimal character $-\kappa^\vee - \rho$ and a Chamber $wX^\star(T)^+_{\mathbb{Q}} \subseteq X^\star(T)^{M,+}_{\mathbb{Q}}$ for $w \in C(\kappa^\vee)^+$. We can ask if this class lifts to a point in $\mathcal{E}^!_{w_{0,M}ww_0}$. This is indeed the case under the assumption of proposition 6.46.

6.9. **Improved slope bounds for interior cohomology and applications.** Using the interior eigenvariety we are able to prove that the conjectured slopes bounds 6.33, 5.29, and 5.10.2 hold for (modified) interior cohomology. The point is that classical points in regular weight satisfy the correct slope bound by corollary 5.44, and these points are dense in the interior eigenvariety.

Theorem 6.48. (1) *Fix* $w \in {}^MW$ *and a weight* $\nu : T^c(\mathbb{Z}_p) \to \mathbb{C}_p^\times$. *For any character* λ *of* $T^\pm$ *on* $\overline{\mathrm{H}}^{\ell_\pm(w)}_{w,an}(K^p,\nu)^{\pm,fs}$ *we have* $v(\lambda) \geq 0$ *in the* $+$ *case and* $v(\lambda) \leq 0$ *in the* $-$ *case.*

(2) *Fix* $w \in {}^MW$, $\kappa \in X^\star(T^c)^{M,+}$, *and a finite order character* $\chi : T^c(\mathbb{Z}_p) \to \overline{F}^\times$. *For any character* λ *of* T^- *on* $\overline{\mathrm{H}}^{\ell_-(w)}_{w}(K^p,\kappa,\chi)^{-,fs}$ *we have* $v(\lambda) \leq w^{-1}(\kappa+\rho)-\rho$.

(3) *Fix* $\kappa \in X^\star(T^c)^{M,+}$ *and a finite order character* $\chi : T^c(\mathbb{Z}_p) : T^c(\mathbb{Z}_p) \to \overline{F}^\times$. *Let* $\nu = -w^{-1}w_{0,M}(\kappa+\rho)-\rho$ *for any* $w \in C(\kappa)^+$. *Then for any character* λ *of* $T\pm$ *on* $\overline{\mathrm{H}}^i(K^p,\kappa,\chi)^{\pm,fs}$ *we have* $v(\lambda) \geq -\nu$ *in the* $+$ *case and* $v(\lambda) \leq -w_0\nu$ *in the* $-$ *case.*

Proof. The first point implies the second point by theorem 6.42 (3). The second point implies the third point as in the proof of proposition 5.42 using corollary 5.27.

We now prove the first point. The eigensystem λ gives a point $(\nu,\lambda_p,\lambda^S)$ of $\mathcal{E}^!$ (where $\lambda_p = \lambda$ in the $+$ case and λ^t in the $-$ case.) We can find another point $(\nu',\lambda'_p,\lambda'^S)$ which satisfies $v(\lambda'_p) = v(\lambda_p)$ and $\nu' = \nu'_{alg}\chi$ is locally algebraic, and $\nu'_{alg} \in X^\star(T^c)^+$ is sufficiently large so that $v(\lambda'_p)$ satisfies $+, sss(\nu)$. Then this point is classical by theorem 6.41 (2), and so the slope bound is satisfied by corollary 5.44. ☐

As a consequence we deduce a vanishing theorem for interior cohomology which improves on theorem 5.69.

Theorem 6.49. *Let* $\kappa \in X^\star(T^c)^{M,+}$ *and let* $\chi : T^c(\mathbb{Z}_p) \to F^\times$ *be a finite order character. We have that* $\overline{\mathrm{H}}^i(K^p,\kappa,\chi)^{ss^M(\kappa)}$ *is supported in the range* $[\ell_{\min}(\kappa),\ell_{\max}(\kappa)]$.

Proof. Theorem 6.48 (2) implies that the complex $\overline{\mathcal{C}ous}(K^p,\kappa,\chi)^{-,ss^M(\kappa)}$ is concentrated in the range $[\ell_{\min}(\kappa),\ell_{\max}(\kappa)]$, which implies the theorem by corollary 5.27

☐

6.10. **Application: local-global compatibility.** In [FP19], section 9, we defined a certain class of cupsidal automorphic forms for the group GL_n/L where L is either a totally real or CM number field. These are called weakly regular, odd, essentially conjugate self dual algebraic cuspidal automorphic representations. Regular essentially conjugate self dual algebraic cuspidal automorphic representations have been studied intensively. They occur in the Betti cohomology of Shimura varieties and one can attach to them compatible systems of Galois representations which satisfy all the expected properties. (see, e.g., [CH13], [BLGGT14]).

Theorem 6.50 (Bellaiche, Caraiani, Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin, Taylor, ...)**.** *Let* π *be a regular, algebraic, (essentially) conjugate self dual cuspidal automorphic representation of* GL_n/L. *In particular* $\pi^c = \pi^\vee \otimes \chi$ *and the infintesimal character of* π *is* $\lambda = ((\lambda_{1,\tau},\cdots,\lambda_{n,\tau})_{\tau\in J})$ *with* $\lambda_{1,\tau} > \cdots > \lambda_{n,\tau}$. *Let* $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$. *There is a continuous Galois representation* $\rho_{\pi,\iota} : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ *such that:*

(1) $\rho^c_{\pi,\iota} \simeq \rho^\vee_{\pi,\iota} \otimes \epsilon_p^{1-n} \otimes \chi_\iota$ *where* χ_ι *is the p-adic realization of* χ *and* ϵ_p *is the cyclotomic character,*

(2) $\rho_{\pi,\iota}$ *is pure,*

(3) $\rho_{\pi,\iota}$ *is de Rham at all places dividing* p*, with* $\iota^{-1}\circ\tau$*-Hodge-Tate weights:* $(-\lambda_{n,\tau}+\frac{n-1}{2},\cdots,-\lambda_{1,\tau}+\frac{n-1}{2})$,

(4) *for all finite place* v *one has:*

$$WD(\rho_{\pi,\iota}|_{G_{F_v}})^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$

Weakly regular, odd, essentially conjugate self dual algebraic cuspidal automorphic representations form a larger class of automorphic representations where one relaxes the regularity assumption to weakly regular odd. Those automorphic representations realize in the coherent cohomology of Shimura varieties. One can attach to them compatible system of Galois representations, but at present their local properties are still mysterious. The techniques of this paper allows for a new construction of the associated Galois representation via analytic families. The advantage of this construction is that we can prove some instances of local-global compatibility for those Galois representations at primes dividing the residue characteristic of the coefficients, using results of Kisin [Kis03] on the interpolation of crystalline periods in analytic families, as for example in the work of Jorza and Mok [Jor12], [Mok14].

Theorem 6.51. *Let* π *be a weakly regular, algebraic, odd, (essentially) conjugate self dual, cuspidal automorphic representation of* GL_n/L*. In particular,* $\pi^c=\pi^\vee\otimes\chi$*. Let* $\lambda=(\lambda_{i,\tau}, 1\leq i\leq n, \tau\in\mathrm{Hom}(L,\overline{\mathbb{Q}}))$ *and* $\lambda_{1,\tau}\geq\cdots\geq\lambda_{n,\tau}$ *be the infinitesimal character of* π*. There is a continuous Galois representation* $\rho_{\pi,\iota}: G_L\to\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ *such that:*

(1) $\rho^c_{\pi,\iota}\simeq\rho^\vee\otimes\epsilon_p^{1-n}\otimes\chi_\iota$ *where* χ_ι *is the* p*-adic realization of* χ*,*

(2) $\rho_{\pi,\iota}$ *is unramified at all finite places* $v\nmid p$ *for which* π_v *is unramified and one has:*

$$WD(\rho_{\pi,\iota}|_{G_{L_v}})^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$

(3) $\rho_{\pi,\iota}$ *has generalized Hodge–Tate weights* $(-\lambda_{n,\tau}+\frac{n-1}{2},\cdots,-\lambda_{1,\tau}+\frac{n-1}{2})$.

(4) *Let* $v\mid p$ *be a place of* L *and assume that* π_v *is a regular principal series representation. Then* $\rho_{\pi,\iota}|_{G_{L_v}}$ *is potentially crystalline and*

$$WD(\rho_{\pi,\iota}|_{G_{L_v}})^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$

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