(Overconvergent) Eichler-Shimura (Ju-Jeng | H. Diao | G. Rosso)

See also Andreatta-Iovita-Stevens | Chojecki-Hansen-Johannsen

§ Introduction

$N > 3$, $p > 3$, $p \nmid N$, $\Gamma = \Gamma_1(N) = \left\{ \gamma \in GL_2(\mathbb{A}_f^p) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$

associate X_Γ compactified modular curve of level Γ (over $\mathbb{Q}$)

$X_\Gamma \supseteq Y_\Gamma$: affine

Want to understand

$$H^0(X_\Gamma(\mathbb{C}), \underline{\omega}^k) \oplus \overline{H^0(X_\Gamma(\mathbb{C}), \underline{\omega}^k_{cusp})} \simeq H^1(Y_\Gamma(\mathbb{C}), \mathrm{Sym}^{k-2}\mathbb{C}^2)$$

Betti cohomology

$\underline{\omega}$ = \omega

as Hecke modules (thm of Eichler-Shimura)

Is there an algebraic version?

Yes (Faltings)

Hecke and Galois modules

$$H^0(X_{\Gamma/\mathbb{Q}_p}, \underline{\omega}^k) \otimes \mathbb{C}_p \oplus H^1(X_{\Gamma/\mathbb{Q}_p}, \underline{\omega}^{-k}) \otimes \mathbb{C}_p(k+1) \simeq H^1_{et}(Y_{\Gamma/\overline{\mathbb{Q}}_p}, \mathrm{Sym}^{k-2}\mathbb{Q}_p) \otimes \mathbb{C}_p(1)$$

Use BGG resolution to compute the de Rham coho, then use étale-de Rham comparison

Ques: Can we p-adically iterate this iso?

Steps: (1) p-adic iteration of the modular sheaf

- Andreatta-Iovita-Stevens, Pilloni, A-I-P,
- AI theory of vector bundles with marked sections
- perfectoid methods $\longleftarrow$ What we'll see today

(2) Overconvergent coho

- Stevens, Ash-Stevens, Urban, Hansen

(3) The morphism

§ p-adic Variation of modular sheaf (perfectoid method)

$$\begin{tikzcd}
\mathcal{X}_{\Gamma(p^\infty)} \arrow[d] \arrow[dd, bend right] & (E, \psi_\Gamma, \varphi_{p^\infty} : T_pE \xrightarrow{\sim} \mathbb{Z}_p^{\times}) \arrow[l, mapsto] \\
\mathcal{X}_{\Gamma(p^n)} \arrow[d] & (E, \psi_\Gamma, \varphi_{p^n} : E[p^n] \xrightarrow{\sim} \mathbb{Z}/p^n) \arrow[l, mapsto] \\
\mathcal{X}_{\Gamma_0(p)} \arrow[d] & (E, \psi_\Gamma, C \subseteq E[p] \text{ of order } p) \arrow[l, mapsto] \\
\mathcal{X} & (E, \psi_\Gamma) \arrow[l, mapsto]
\end{tikzcd}$$

perfectoid space

$\Gamma(p^n) = \ker\left(Gl_2(\mathbb{Z}_p) \to Gl_2(\mathbb{Z}/p^n)\right)$

$\Gamma_0(p) = \left\{ \gamma \in Gl_2(\mathbb{Z}_p) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p \right\}$

$\mathcal{X}$ = adic space assoc to $X_\Gamma / \mathbb{C}_p$

* The purple map is called h_{Iwahori} (h_{Iw})

Over the perfectoid space

$$\begin{array}{ccc}
\mathcal{X}_{\Gamma(p^\infty)} & \xrightarrow{\pi_{HT}} & \mathbb{P}^1(\mathbb{C}_p) \\
(E, \psi_\Gamma, \varphi_{p^\infty}) & \longmapsto & (0 \hookrightarrow \operatorname{Lie} E \hookrightarrow \underbrace{T_pE \otimes \mathbb{C}_p}_{\cong\, \mathbb{C}_p^2})
\end{array}$$

facts: (i) We have $Gl_2(\mathbb{Z}_p)$ - actions on both sides

π_{HT} is $Gl_2(\mathbb{Z}_p)$- equiv

(ii) If E is ord, $\Rightarrow \pi_{HT}(E) \in \mathbb{P}^1(\mathbb{Q}_p)$

Define : $\mathbb{P}^1_w = \left\{ (1 : z) \in \mathbb{P}^1(\mathbb{C}_p) : \inf\{|z - s| : s \in \mathbb{Z}_p\} \leq |p^w| \right\}$

for any $w \in \mathbb{Q}_{>0}$

Defn: (1) We say E is <u>w-ordinary</u> if $\pi_{HT}(E) \in \mathbb{P}^1_w$

E is ordinary $\Leftrightarrow$ w-ord $\forall w$

(2) $\mathcal{X}_{\infty, w} := \pi_{HT}^{-1}(\mathbb{P}^1_w)$

$\mathcal{X}_{\text{Iwahori}, w} := h_{\text{Iw}}(\mathcal{X}_{\infty, w}) \subseteq \mathcal{X}_{\Gamma_0(p)}$

fact:

$\mathbb{P}^1_w \circlearrowleft \Gamma_0(p)$ by $(1 : z)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(1 \quad (a + zc)^{-1}(b + zd)\right)$

* Some dependence on w maybe

$\Gamma_0(p) \curvearrowright \mathcal{X}_{\infty, w}$ & "$\mathcal{X}_{\text{Iw}, w} = \mathcal{X}_{\infty, w} / \Gamma_0(p)$"

Define $\mathfrak{z} := \pi_{HT}^* z$ on $\mathcal{X}_{\infty,w}$

Take char $\underline{\kappa} : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times$ (cont). We say $\underline{\kappa}$ is <u>w-analytic</u> if it extends to $\mathbb{Z}_p^\times(1+p^w\mathcal{O}_{\mathbb{C}_p})$

Given w-analytic $\underline{\kappa}$ & $(a,c) \in \mathbb{Z}_p^\times \times p\mathbb{Z}_p$, define

$$\underline{\kappa}(a+\mathfrak{z}c) : \mathcal{X}_{\infty,w} \xrightarrow{\pi_{HT}} \mathbb{P}^1_w \longrightarrow \mathbb{C}_p^\times$$

$$\mathfrak{z} \longmapsto z \longmapsto \underline{\kappa}(a+zc)$$

Define <u>w-overconvergent modular sheaf of wt $\underline{\kappa}$</u> is a subsheaf $\underline{\omega}_w^{\kappa} \subset h_{I_w,*}\mathcal{O}_{\mathcal{X}_{\infty,w}}$ consisting of sections f st

$$\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p), \quad \gamma \cdot f = \underline{\kappa}(a+\mathfrak{z}c)^{-1} f$$

Prop (1) for any w-analytic $\underline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, we have

$$H^0(\mathcal{X}_{I_w,w}, \underline{\omega}_w^{k}) \hookrightarrow M_{\underline{\kappa}}^{p\text{-adic}}(\Gamma)$$

(2) The U_p operator acts compactly on $H^0(\mathcal{X}_{I_w,w}, \underline{\omega}_w^{k})$

(3) For any $k \in \mathbb{Z}_{\geq 0}$, we have $H^0(\mathcal{X}_{\Gamma_0(p)}, \underline{\omega}^{k}) \xrightarrow{\text{res}} H^0(\mathcal{X}_{I_w,w}, \underline{\omega}_w^{k})$

classical modular form of wt k on $\mathcal{X}_{\Gamma_0(p)}$

§ Overconvergent cohomology

$T_0 := \mathbb{Z}_p^\times \times p\mathbb{Z}_p$ ← can think $\forall (a,c) \in T_0$ $\exists\, b,d \in \mathbb{Z}_p$ st $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$

- $\mathbb{Z}_p^\times$ acts on T_0 by multiplication
- $\Xi := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap GL_2(\mathbb{Q}_p)$ acts on T_0 via left mult

$r \in \mathbb{Q}_{>0}$, $\underline{\kappa} : \mathbb{Z}_p^\times \to L$, r-analytic char, $L/\mathbb{Q}_p$ = fin

$$A_{\underline{\kappa}}^r(T_0, L) = \{ f : T_0 \to L : f \text{ is r-analytic}, \ f(a\alpha, c\alpha) = \underline{\kappa}(\alpha) f(a,c) \ \forall ((a,c),\alpha) \in T_0 \times \mathbb{Z}_p^\times \}$$

r-analytic funcns on T_0 of wt $\underline{\kappa}$

$D^r_{\underline{k}}(T_0, L) = \mathrm{Hom}^{cont}_L\left(A^r_{\underline{k}}(T_0, L), L\right)$ = r-analytic distributions

The left action of Ξ on T_0 gives a left action on $D^r_{\underline{k}}(T_0, L)$

fact: (1) $D^r_{\underline{k}}(T_0, L)$ is a Banach space

So, we can define $D^{r,0}_{\underline{k}}(T_0, L)$ a unit ball

(2) $\exists$ filtration $\mathrm{Fil}^{\gamma} D^{r,0}_{\underline{k}}(T_0, L)$ st $\left\{ D^{r,0}_{\underline{k}}(T_0, L)/\mathrm{Fil}^{\gamma} \right\}_{\gamma}$ forms a proj system of Ab gps and

$$\varprojlim(\quad) \simeq D^{r,0}_{\underline{k}}$$

$\Rightarrow$ we have a proj sys of loc constt sheaves of fin ab gps on $\mathcal{X}_{\Gamma_0(p), \mathrm{K\acute{e}t}}$ $\left\{ \mathcal{D}^{r,0}_{\underline{k},j} \right\}_j$

$\rightsquigarrow$ can consider $H^1_{\mathrm{K\acute{e}t}}(\mathcal{X}_{\Gamma_0(p)}, \mathcal{D}^r_{\underline{k}}) := \varprojlim_j H^1_{\mathrm{K\acute{e}t}}(\quad, \mathcal{D}^{r,0}_{k,\gamma})[1/p]$

fact: Can define Hecke operators on $H^1_{\mathrm{K\acute{e}t}}(\mathcal{X}_{\Gamma_0(p)}, \mathcal{D}^r_{\underline{k}})$

and U_p acts compactly $\Rightarrow$ "slope decomposition"

Prop: When $\underline{k} = k \in \mathbb{Z}_{>0}$, we have a natural equiv map

$$H^1_{\mathrm{K\acute{e}t}}(\mathcal{X}_{\Gamma_0(p)}, \mathcal{D}^r_{\underline{k}}) \to H^1_{\mathrm{K\acute{e}t}}(\mathcal{X}_{\Gamma_0(p)}, \mathrm{Sym}^k \mathbb{Q}_p^2)$$

Thm: (Steven's Control Thm)

$0 < h < k-1$, then $H^1_{\mathrm{K\acute{e}t}}(\mathcal{X}_{\Gamma_0(p)}, \mathcal{D}^r_{\underline{k}})^{\leq h} \xrightarrow{\sim} H^1_{\mathrm{K\acute{e}t}}(\mathcal{X}_{\Gamma_0(p)}, \mathrm{Sym}^k \mathbb{Q}_p^2)^{\leq h}$

§ Overconvergent Eichler-Shimura

$\mathcal{X}_{Iw,w,\text{pro-K\'et}}$ = "limit of fin Kummer-étale towers" $\supset \mathcal{X}_{\infty,w}$

$\downarrow \nu$

$\mathcal{X}_{Iw,w,\mathrm{K\acute{e}t}} \leftarrow$ "$\mathcal{D}^r_{\underline{k}}$"

$\downarrow$

$\mathcal{X}_{Iw,w} \leftarrow \underline{\omega}^{\underline{k}}_w$ (sheaf)

can do completion here on the level of sheaves note: We have two kinds of Structure sheaf on $\mathcal{X}_{Iw,w,\text{prokét}}$

$\mathcal{O}_{\mathcal{X}_{Iw,w,\text{prokét}}}$, $\mathcal{O}^+_{\mathcal{X}_{Iw,w,\text{prokét}}}$ and their completed versions $\hat{\mathcal{O}}, \hat{\mathcal{O}}^+$

Define

(i) $\mathcal{O}\mathcal{D}^r_{\underline{\kappa}} := \left(\varprojlim_{\nu} \nu^{-1} \mathcal{D}^{r,0}_{\underline{\kappa},r} \otimes_{\mathbb{Z}_p} \hat{\mathcal{O}}^+\right)[1/p]$

(ii) $\hat{\underline{\omega}}^{\kappa}_w := \underline{\omega}^{\kappa}_w \otimes_{\mathcal{O}_{\mathcal{X}_{Iw,w}}} \hat{\mathcal{O}}$

They are both $\hat{\mathcal{O}}$ modules

Prop: (1) $H^1_{két}(\mathcal{X}_{\Gamma_0(p)}, \mathcal{D}^r_{\underline{\kappa}}) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p \simeq H^1_{pro}(\mathcal{X}_{\Gamma_0(p)}, \mathcal{O}\mathcal{D}^r_{\underline{\kappa}})$

(2) We have a natural Hecke & Galois-equiv morphism

$$H^1_{pro}(\mathcal{X}_{Iw,w,pro}, \hat{\underline{\omega}}^{\kappa}_w) \to H^0(\mathcal{X}_{Iw,w}, \underline{\omega}^{k+2}_w)(-1)$$

The Overconvergent Eichler-Shimura is constructed as follows:

1. Look at the localised site $\mathcal{X}_{Iw,w,pro} / \mathcal{X}_{\infty,w}$

$$\eta_{\underline{\kappa}} : \mathcal{O}\mathcal{D}^r_{\underline{\kappa}} \to \hat{\underline{\omega}}^{\kappa}_w$$
$$\mu \otimes a \mapsto \mu(\underline{\kappa}(a+\underline{z}c)) \otimes a$$

Check: $\eta_{\underline{\kappa}}$ is Ξ - equiv

2. Take $\Gamma_0(p)$ - invariants $\rightsquigarrow$ check Hecke equiv

3. Take cohomology

$$\begin{array}{ccc}
H^1(\mathcal{X}_{Iw,w,pro}, \mathcal{O}\mathcal{D}^r_{\underline{\kappa}}) & \to & H^1_{pro}(\mathcal{X}_{Iw,w,pro}, \hat{\underline{\omega}}^{\kappa}_w) \\
\uparrow & & \downarrow \\
H^1_{két}(\mathcal{X}_{Iw,w}, \mathcal{D}^r_{\underline{\kappa}}) \hat{\otimes} \mathbb{C}_p & & H^0(\mathcal{X}_{Iw,w}, \underline{\omega}^{k+2}_w)(-1) \\
\uparrow \text{res} & & \\
H^1_{két}(\mathcal{X}_{\Gamma_0(p)}, \mathcal{D}^r_{\underline{\kappa}}) \hat{\otimes} \mathbb{C}_p & &
\end{array}$$

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