GENERALIZED MAZUR'S GROWTH NUMBER CONJECTURE

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Abstract. Let F be a totally real field. Let A be a simple modular self-dual abelian variety defined over F. We study the growth of the corank of Selmer groups of A over $\mathbb{Z}_p$-extensions of a CM extension of F. We propose an extension of Mazur's growth number conjecture for elliptic curves to this new setting. We provide evidence supporting an affirmative answer by studying special cases of this problem, generalizing previous results on elliptic curves and imaginary quadratic fields.

1. Introduction

Let $p \neq 2$ be a fixed prime number. Let A be a simple modular self-dual abelian variety over a totally real field F, so that F is isomorphic to a subfield of $\mathrm{End}(\mathsf{A}) \otimes \mathbb{Q}$, with $d = \dim(\mathsf{A}) = [F : \mathbb{Q}]$. In particular, A is a simple quotient of the Jacobian of a Shimura curve over F, corresponding to a Hilbert modular form. Let σ_{A} denote the weight 2 and level $\mathfrak{N}$ cuspidal automorphic representation attached to A. Since A is self-dual, the central character of σ_{A} is trivial.

Throughout, we assume the following hypothesis holds.

(**ORD**) A has *potentially* good ordinary reduction at all primes above p.

Let K be a CM extension of F and suppose that the Leopoldt conjecture is satisfied, i.e., the compositum of all $\mathbb{Z}_p$-extensions of K, which we denote by K_∞, is a $(d+1)$-dimensional abelian p-adic Lie group. Let $\Gamma_\infty = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p^{\oplus(d+1)}$.

Let $\mathfrak{p} \mid p$ be a fixed prime of F, which induces an embedding $\iota_{\mathfrak{p}} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. The ring of integers of the completion $F_{\mathfrak{p}}$ is denoted by $\mathcal{O}_{\mathfrak{p}}$. We are interested in the growth of the $\mathcal{O}_{\mathfrak{p}}$-corank of $\mathfrak{p}$-primary Selmer groups of A over $\mathbb{Z}_p$-extensions of K. When $F = \mathbb{Q}$ and A is an elliptic curve, it is predicted by B. Mazur [Maz83, §18] that the Selmer coranks of A over finite extensions of K inside a $\mathbb{Z}_p$-extension should be bounded, except possibly the anticyclotomic $\mathbb{Z}_p$-extension when A/K has root number -1. More generally, in [MR03, Question 2.13], B. Mazur–K. Rubin asked whether it might be possible to use towers of Heegner points in Shimura curves over totally real fields to account for (at least some of) the expected Mordell–Weil growth as one ascends the finite intermediate extensions of the anti-cyclotomic hyperplane.

Our results are inspired by the aforementioned question raised by Mazur–Rubin, the recent developments on the p-adic Gross–Zagier formula on Shimura curves by D. Disegni [Dis17], and the works of J. Nekovář [Nek07, Nek08] which show that growth in the anticyclotomic direction can indeed be accounted for by Heegner points. More precisely, we study the following analogue of Mazur's Growth Number Conjecture.

Growth Number Problem. *Fix a prime $p \neq 2$. Let A be a simple modular self-dual abelian variety over a totally real field F with potentially good ordinary reduction at $\mathfrak{p} \mid p$. Let K/F be a CM extension and $\mathcal{K}/K$ be a $\mathbb{Z}_p$-extension. Write $\mathcal{K}_n$ to denote the unique subfield of degree p^n of K in $\mathcal{K}$. Then for $n \gg 0$*

$$\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}} \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n) = cp^n + O(1),$$

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where the growth number c *is given by*

$$c = \begin{cases} 0 & \text{if } \mathcal{K} \not\subseteq K_{\mathrm{ac}} \text{ or } (\mathsf{A}, K) \text{ has 'sign' } +1 \\ 1 & \text{if } \mathcal{K} \subseteq K_{\mathrm{ac}}, (\mathsf{A}, K) \text{ has 'sign' } -1 \text{ and is 'generic'} \\ 2 & \text{if } \mathcal{K} \subseteq K_{\mathrm{ac}}, (\mathsf{A}, K) \text{ has 'sign' } -1 \text{ and is 'exceptional'.} \end{cases}$$

Here, K_{ac} denotes the compositum of all anticyclotomic $\mathbb{Z}_p$-extensions of K and $\mathrm{Gal}(K_{\mathrm{ac}}/K) \simeq \mathbb{Z}_p^{\oplus d}$. *The pair (A, K) is called 'generic' if A has no complex multiplication (CM), or the CM field of A is different from K. On the other hand, the pair (A, K) is called 'exceptional' if A has CM by (an order in) K. The* sign *of (A, K) is defined analogous to* [Maz83, § 6]*) to mean*

- *the sign of the functional equation of $L(\mathsf{A}/K, s)$ in the 'generic' case* and
- *the sign of the functional equation of $L(\varphi, s)$ in the 'exceptional' case, where φ is the Hecke character of K satisfying*
$$L(\mathsf{A}/F, s) = L(\varphi, s).$$

1.1. **Progress towards this problem.** As per our knowledge, the Growth Number Problem has been previously studied mainly in the case when $F = \mathbb{Q}$ and A is an elliptic curve. The original conjecture is completely settled for CM elliptic curves in the 'exceptional case'; see [Gre99, Theorems 1.7 and 1.8]. In a recent preprint by H. Li–R. Xu (see [LX25]), the authors study the growth of Mordell–Weil ranks of CM abelian varieties associated with Hecke characters of infinite type $(1, 0)$ over an imaginary quadratic field K along the anticyclotomic $\mathbb{Z}_p$-extension of K.

In the last few years, there is increased interest in understanding the original conjecture of Mazur for non-CM elliptic curves. In particular, we mention the following articles [KMS23, GHKL25, KL25]. Each of them takes a different approach to solve special cases of the problem, but somewhat surprisingly a solution is provided only when the Mordell–Weil rank of the elliptic curve over K is at most 1. These results also focus on the case that p is a prime of good ordinary reduction, which is precisely the setting in which Mazur first formulated his conjecture.

1.2. **Main Result I.** The first result of this article gives an affirmative answer to the Growth Number Problem under the generalized Heegner hypothesis:

(**GHH**$^+$) $\epsilon_{K/F}(\mathfrak{N}) = (-1)^{d-1}$, where $\epsilon_{K/F}$ is the quadratic character attached to the extension K/F. Furthermore, all primes of F lying above p split into two distinct primes in K.

The proof is analytic in nature, relying on Heegner points attached to A over K and their relations with the derivative of a multi-variable $\mathfrak{p}$-adic L-function (see §3). Furthermore, we assume one inclusion of the Iwasawa main conjecture of A over K_∞ holds, which allows us to study the growth of the Selmer corank via the analytic $\mathfrak{p}$-adic L-function. It generalizes our previous result [KL25, Theorem A] wherein we proved Mazur's Growth Number Conjecture in $\mathbb{Z}_p$-extensions of an imaginary quadratic field K for non-CM elliptic curves $\mathsf{E}/\mathbb{Q}$ at primes $p \geq 5$ of good ordinary reduction under (comparable) technical hypotheses.

Theorem A (Theorems 5.3 and 5.4)**.** *Fix a prime $p \neq 2$. Let A be a simple modular self-dual abelian variety of* GL_2*-type over a totally real field F with trivial central character satisfying* (**ORD**)*. Let K/F be a CM extension such that $\mathsf{A}(K)[\mathfrak{p}]$ is trivial and that the $\mathfrak{p}$-primary Selmer group* $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})$ *is a co-torsion Λ_{cyc}-module. Suppose that* (**GHH**$^+$) *and one inclusion of the Iwasawa main conjecture for A over K_∞ holds; see* (**h-IMC**)*. Writing z_{Heeg} to denote the Heegner point of A over K, if the p-adic height $\langle z_{\mathrm{Heeg}}, z_{\mathrm{Heeg}} \rangle_K$ is nonzero, then the Growth Number Problem has a positive solution.*

1.3. **Main Result II.** The second result of this article is purely algebraic in nature, where we study the growth of the Selmer corank using characteristic ideals. It generalizes [GHKL25, Theorem A] where Mazur's Growth Number Conjecture for elliptic curves over $\mathbb{Z}_p$-extensions of an imaginary quadratic field K is studied under a hypothesis on the structure of the Selmer group over the unique

$\mathbb{Z}_p^{\oplus 2}$-extension of K. In particular, we extend the aforementioned result to the setting of abelian varieties. Although we follow a line of argument similar to that presented in [GHKL25], we give a (slight) simplification of the result and remove the non-anomalous hypothesis.

In what follows, K_{cyc} denotes the cyclotomic $\mathbb{Z}_p$-extension of K with $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(K_{\mathrm{cyc}}/K)$. Let $\mathcal{O}_{\mathfrak{p}}$ denote the ring of integers of the completion of F at $\mathfrak{p}$ and write Λ_∞ (resp. Λ_{cyc}) for the Iwasawa algebra $\mathcal{O}_{\mathfrak{p}}[\![\Gamma_\infty]\!]$ (resp. $\mathcal{O}_{\mathfrak{p}}[\![\Gamma_{\mathrm{cyc}}]\!]$).

Theorem B (Theorems 6.1 and 6.2)**.** *Fix a prime $p \neq 2$. Let A be a simple modular self-dual abelian variety of GL_2-type over a totally real field F with trivial central character satisfying* (**ORD**)*. Let K/F be a CM extension such that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ is Λ_{cyc}-torsion. Suppose that one of the following conditions hold:*

(*i*) *the order of vanishing of $\mathrm{char}_{\Lambda_{\mathrm{cyc}}}\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ at the trivial character of Γ_{cyc} is 0.*

(*ii*) (**GHH$^+$**) *holds, the order of vanishing of $\mathrm{char}_{\Lambda_{\mathrm{cyc}}}\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ at the trivial character of Γ_{cyc} is 1, and $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ is a direct sum of cyclic Λ_∞-modules.*

Then the Growth Number Problem has a positive answer.

1.4. **Organization.** Including this introduction, the article has six sections. Section 2 is preliminary in nature; we introduce the notation that is used throughout the paper. We remind the reader of some definitions and fundamental results that are used several times in our arguments. In Section 3, we review Disegni's result on the $\mathfrak{p}$-adic L-function attached to A and the relation between its derivative and Heegner points proven in [Dis22]. Furthermore, we carry out calculations on the specialization of the $\mathfrak{p}$-adic L-function to a $\mathbb{Z}_p$-extension of K, reducing its non-vanishing to that of the $\mathfrak{p}$-adic height of z_{Heeg}, which is crucially utilized in the proof of Theorem A. Another key ingredient of this proof is to show that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ admit no non-trivial pseudonull submodule; this is done in Section 4 utilizing the main result of [Gre16] by R. Greenberg. We complete the proof of the theorem in Section 5. In Section 6, we prove Theorem B, providing evidence for the Growth Number Problem using algebraic tools under a slightly different set of hypotheses.

1.5. **Outlook.** As has been pointed out previously, the higher rank case is still out of reach. The supersingular case would also require more work and new ideas. One may hope to utilize results on supersingular abelian varieties [BL17, BL15, Pon20, LP20, IL25] combined with earlier works of elliptic curves [IP06, LL22, LS20, HL20].

When the corank of $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)$ is one, our approach for studying the Growth Number Problem hinges on the results of Disegni and Nekovář; therefore, the setup in which we can answer the question is dictated by their work. In particular, our method of proof cannot be used to address the question of Selmer rank growth in $\mathbb{Z}_p$-extensions of a general CM field K of an abelian variety A that does not have real multiplication by the maximal real subfield of K. On the other hand, if $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)$ is finite, then standard arguments involving (Mazur's) Control Theorem that we use to prove Theorem B under the condition (i) can be adapted readily to obtain an answer.

Let A be an abelian variety over a number field K and $\mathcal{K}/K$ be a $\mathbb{Z}_p$-extension. One can ask about the Selmer corank growth of $\mathrm{Sel}_{p^\infty}(\mathsf{A}/\mathcal{K}_n)$ as $n \to \infty$, where $\mathcal{K}_n$ denotes the n-th layer of $\mathcal{K}/K$. In the context of this paper, when K is a CM field and A has real multiplication by the maximal real subfield of K, we see that

$$\mathrm{Sel}_{p^\infty}(\mathsf{A}/\mathcal{K}_n) = \bigoplus_{\mathfrak{p}|p} \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)$$

under appropriate hypotheses. Thus,

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(\mathsf{A}/\mathcal{K}_n) = cdp^n + O(1),$$

where c is the 'growth number' in the Growth Number Problem and $d = \dim(\mathsf{A})$. One may speculate that such a formula might hold for more general number fields K.

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2. Notation and Preliminaries

2.1. **Iwasawa algebras and projections.** Recall from the introduction that F is a totally real field of degree d and K is a CM extension of F. The compositum of all $\mathbb{Z}_p$-extensions of K is denoted by K_∞ and $\Gamma_\infty = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p^{\oplus(d+1)}$ Let K_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of K and K_{ac} the anticyclotomic extension of K inside K_∞. Write

$$\Gamma_\infty = \overline{\langle \sigma_0, \sigma_1, \dots, \sigma_d \rangle}$$

such that $\mathrm{Gal}(K_{\mathrm{cyc}}/K)$ is generated by σ_0 and $\mathrm{Gal}(K_{\mathrm{ac}}/K)$ is generated by $\sigma_1, \dots, \sigma_d$. This gives rise to the following isomorphism

$$\Lambda_\infty = \mathcal{O}_{\mathfrak{p}}[\![\Gamma_\infty]\!] \simeq \mathcal{O}_{\mathfrak{p}}[\![X_0, \dots, X_d]\!],$$

sending σ_i to $X_i - 1$.

Given a $\mathbb{Z}_p$-extension $\mathcal{K}/K$ with $\Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K}/K)$, we have a natural projection

$$\pi_{\mathcal{K}} : \Gamma_\infty \longrightarrow \Gamma_{\mathcal{K}},$$

whose kernel is isomorphic to $\mathbb{Z}_p^{\oplus d}$. We can write

$$\ker \pi_{\mathcal{K}} = \left\{ \prod_{i=0}^{d} \sigma_i^{c_i} : \sum_{i=0}^{d} a_i c_i = 0 \right\},$$

for some $a_i \in \mathbb{Z}_p$ such that not all a_i are zero. This allows us to identify the set of $\mathbb{Z}_p$-extensions of K with $\mathbb{P}^d(\mathbb{Z}_p)$. After scaling if necessary, we may assume that $c_j \in \mathbb{Z}_p^\times$ for some $j = j(\mathcal{K})$. Then, for all $i \in \{0, \dots, d\}$,

$$\pi_{\mathcal{K}}(\sigma_i) = \pi_{\mathcal{K}}\left(\sigma_j^{\frac{c_i}{c_j}}\right).$$

In particular, we see that $\Gamma_{\mathcal{K}}$ is topologically generated by $\pi_{\mathcal{K}}(\sigma_{j(\mathcal{K})})$. We shall denote this element by $\sigma_{\mathcal{K}}$ and write $X_{\mathcal{K}} = \sigma_{\mathcal{K}} - 1$.

Set $\Lambda_{\mathcal{K}} = \mathcal{O}_{\mathfrak{p}}[\![\Gamma_{\mathcal{K}}]\!] \cong \mathcal{O}_{\mathfrak{p}}[\![X_{\mathcal{K}}]\!]$. The natural extension of $\pi_{\mathcal{K}}$ to $\Lambda_\infty \to \Lambda_{\mathcal{K}}$ (which we still denote by the same symbol) can be realized as

$$\begin{aligned} \pi_{\mathcal{K}} : \mathcal{O}_{\mathfrak{p}}[\![X_0, \dots, X_d]\!] &\longrightarrow \mathcal{O}_{\mathfrak{p}}[\![X_{\mathcal{K}}]\!], \\ f(X_0, \dots, X_d) &\mapsto f\left((1 + X_{\mathcal{K}})^{\frac{c_0}{c_{j(\mathcal{K})}}} - 1, \dots, (1 + X_{\mathcal{K}})^{\frac{c_d}{c_{j(\mathcal{K})}}} - 1\right). \end{aligned}$$

Therefore,

$$\frac{\mathrm{d}\pi_K(f)}{\mathrm{d}X_{\mathcal{K}}} = \sum_{i=0}^{d} \frac{c_i}{c_{j(\mathcal{K})}} (1 + X_{\mathcal{K}})^{\frac{c_i}{c_{j(\mathcal{K})}} - 1} \frac{\partial f}{\partial X_i} \left((1 + X_{\mathcal{K}})^{\frac{c_0}{c_{j(\mathcal{K})}}} - 1, \dots, (1 + X_{\mathcal{K}})^{\frac{c_d}{c_{j(\mathcal{K})}}} - 1\right),$$

which tells us that

$$\left.\frac{\mathrm{d}\pi_K(f)}{\mathrm{d}X_{\mathcal{K}}}\right|_{X_{\mathcal{K}}=0} = \sum_{i=0}^{d} \frac{c_i}{c_{j(\mathcal{K})}} \cdot \frac{\partial f}{\partial X_i}(0, \dots, 0). \tag{2.1}$$

Note that if $\mathcal{K} = K_{\mathrm{cyc}}$, then $j(K_{\mathrm{cyc}}) = 0$, corresponding to $(1 : 0 : \dots 0) \in \mathbb{P}^d(\mathbb{Z}_p)$. We write π_{cyc} for $\pi_{K_{\mathrm{cyc}}}$, which is given by $f(X_0, X_1, \dots, X_d) \mapsto f(X_0, 0, \dots, 0)$. Furthermore, we write $\Lambda_{\mathrm{cyc}} = \mathcal{O}_{\mathfrak{p}}[\![\Gamma_{\mathrm{cyc}}]\!]$ for the corresponding Iwasawa algebra.

Throughout this article, we often consider $\mathcal{K}$ to be a non-anticyclotomic $\mathbb{Z}_p$-extension of K, i.e. $\mathcal{K} \not\subseteq K_{\mathrm{ac}}$. Such $\mathcal{K}$ corresponds to $(c_0 : \cdots : c_d) \in \mathbb{P}^d(\mathbb{Z}_p)$, where $c_0 \neq 0$.

2.2. **Control Theorems and Rank Growth in $\mathbb{Z}_p$-Extensions of Number Fields.** Let A/F be a simple abelian variety of GL_2-type and level $\mathfrak{N}$ over a totally real field F with potentially good ordinary reduction at all primes above p. Let $\Sigma(F)$ be a finite set of primes in F containing $\mathfrak{p}$ and all primes of bad reduction for A; in other words $\Sigma(F) \supseteq \{\mathfrak{p}\} \cup \{v \colon v \mid \mathfrak{N}\}$. For any field L/F, define $\Sigma(L)$ to be the set of places of L lying above those in $\Sigma(F)$, and write $G_\Sigma(L)$ for the Galois group of the maximal extension of L that is unramified outside of $\Sigma(L)$. Furthermore, for any $v \in \Sigma(F)$ and any finite extension L/F, write

$$J_v(\mathsf{A}/L) = \bigoplus_{w|v} H^1(L_w, \mathsf{A})[\mathfrak{p}^\infty].$$

When $\mathcal{L}/L$ is an infinite extension of L, set

$$J_v(\mathsf{A}/\mathcal{L}) = \varinjlim_{L \subseteq L' \subseteq \mathcal{L}} J_v(\mathsf{A}/L').$$

Definition 2.1. *Let A/F be a simple abelian variety of GL_2-type over a totally real field F with potentially good ordinary reduction at all primes above p. Let $\Sigma(F)$ be any finite set of primes containing those dividing $\mathfrak{p}\mathfrak{N}$. For any extension L/F, define the* Selmer group

$$\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/L) := \ker\left(H^1(G_\Sigma(L), \mathsf{A}[\mathfrak{p}^\infty]) \longrightarrow \prod_{v \in \Sigma} J_v(\mathsf{A}/L)\right).$$

We now recall the statement of Mazur's Control Theorem, which allows us to study the growth behaviour of Selmer groups in $\mathbb{Z}_p$-extensions.

Theorem 2.2 (Mazur's Control Theorem)**.** *Fix an odd prime p. Let F be a totally real field number field and let A/F be an abelian variety of GL_2-type which has* potentially ordinary reduction *at all primes of F lying over p. Let K/F be a quadratic extension of F which is a CM field. Let $\mathcal{K}$ be any $\mathbb{Z}_p$-extension of K and $\mathcal{K}_n$ denote the n-th layer of this extension with $\mathrm{Gal}(\mathcal{K}_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$. Then the kernel and cokernel of the natural map*

$$\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n) \longrightarrow \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^{\mathrm{Gal}(\mathcal{K}/\mathcal{K}_n)}$$

are finite and of bounded order independent of n.

Proof. This is proved in the same way as [Gre03, Proposition 5.1]; see also [MO06, Theorem 2]. □

Corollary 2.3. *Suppose that the assumptions of Theorem 2.2 hold. Set $r = \mathrm{corank}_{\Lambda_{\mathcal{K}}}(\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}))$. Then as $n \to \infty$,*

$$\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}}(\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)) = rp^n + O(1).$$

In particular, if $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)$ is finite then as $n \to \infty$,

$$\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}}(\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)) = O(1).$$

Proof. The arguments for both assertions are standard. They are recorded in [Gre99, Corollaries 4.9 and 4.12] for $\mathbb{Z}_p$-coranks of p^∞-Selmer groups of elliptic curves, but the proofs go through for the current setting. □

3. $\mathfrak{p}$-adic L-functions

Throughout this section, we assume (**GHH$^+$**) holds. We review results of Disegni [Dis17] that will be utilized in our proof of Theorem A. For each place of F lying above p, we fix a level 0 additive character (see the end of p. 1993 in *op. cit.*).

Theorem 3.1 ([Dis17, Theorem A]). *Let $\mathfrak{p}$ be a place of F lying above p where A has potentially good ordinary reduction. There exists a unique $\mathfrak{p}$-adic L-function*[1] $L_{\mathfrak{p}}(\mathsf{A}) \in F_{\mathfrak{p}} \otimes \Lambda_\infty$ *such that for all finite characters χ of Γ_∞,*

$$L_{\mathfrak{p}}(\mathsf{A})(\chi) = C_\chi \cdot \frac{L(\mathsf{A}/K, \chi, 1)}{\Omega}$$

for some constant C_χ and a period Ω. Here, $\dfrac{L(\mathsf{A}/K, \chi, 1)}{\Omega}$ is an algebraic number, which is regarded as an element of $\overline{\mathbb{Q}_p}$ through $\iota_{\mathfrak{p}}$.

The sign of the functional equation of $L(\mathsf{A}, K, \chi, s)$ at $s = 1$ is constant for all finite characters χ of $\mathrm{Gal}(K_{\mathrm{ac}}/K)$; see discussion in [Dis17, top of p. 1999]. In particular, since A is assumed to be self-dual, $L_{\mathfrak{p}}(\mathsf{A})(\chi) = 0$ for all such χ under our running hypotheses. If we consider $L_{\mathfrak{p}}(\mathsf{A})$ as a power series in $X_0, X_1, \ldots, X_d$, we have $L_{\mathfrak{p}}(\mathsf{A})(0, X_1, \ldots, X_d) = 0$. Consequently, we can expand $L_{\mathfrak{p}}(\mathsf{A})$ as a power series in X_0 with coefficients in $\Lambda_{\mathrm{ac}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $\Lambda_{\mathrm{ac}} := \mathcal{O}_{\mathfrak{p}}[\![\mathrm{Gal}(K_{\mathrm{ac}}/K)]\!] = \mathcal{O}_{\mathfrak{p}}[\![X_1, \ldots, X_d]\!]$. More specifically, we have

$$L_{\mathfrak{p}}(\mathsf{A}) = G_{\mathfrak{p}}(\mathsf{A}) X_0 + O(X_0^2), \tag{3.1}$$

where $G_{\mathfrak{p}}(A) \in \Lambda_{\mathrm{ac}} \otimes \mathbb{Q}_p$.

Proposition 3.2. *Let $\mathcal{K}$ be a non-anticyclotomic $\mathbb{Z}_p$-extension of K. If $G_{\mathfrak{p}}(\mathsf{A})(0, \ldots, 0) \neq 0$, then*

$$\left.\frac{\mathrm{d}\pi_K(L_{\mathfrak{p}}(\mathsf{A}))}{\mathrm{d}X_{\mathcal{K}}}\right|_{X_{\mathcal{K}}=0} \neq 0.$$

Proof. Recall that $\mathcal{K}$ corresponds to a $\mathbb{Z}_p$-extension such that $(c_0 : \cdots : c_d) \in \mathbb{P}^d(\mathbb{Z}_p)$ with $c_0 \neq 0$. It follows from (2.1) and (3.1), that

$$\left.\frac{\mathrm{d}\pi_K(L_{\mathfrak{p}}(\mathsf{A}))}{\mathrm{d}X_{\mathcal{K}}}\right|_{X_{\mathcal{K}}=0} = \frac{c_0}{c_{j(\mathcal{K})}} \cdot G_{\mathfrak{p}}(\mathsf{A})(0, \ldots, 0) + \sum_{i=1}^{d} \frac{c_i}{c_{j(\mathcal{K})}} \cdot \left(\frac{\partial G_{\mathfrak{p}}(A)}{\partial X_i} X_0\right)(0, \ldots, 0),$$

which is a non-zero multiple of $G_{\mathfrak{p}}(\mathsf{A})(0, \ldots, 0)$. Hence, the lemma follows. □

By [Dis17, Theorem C(4)], $G_{\mathfrak{p}}(\mathsf{A})$ is described via the Λ_{ac}-adic heights of the Heegner points on A defined over finite sub-extensions of K_{ac}/K. In particular, $G_{\mathfrak{p}}(\mathsf{A})(0, \ldots, 0)$ is a non-zero multiple of the $\mathfrak{p}$-adic height of the Heegner point z_{Heeg} attached to A/K. The reader is referred to [Dis17, §1.1] for the definition of z_{Heeg} (where the character χ in *loc. cit.* is taken to be the trivial character).

We use the notation $\langle -, - \rangle_K$ to denote the p-adic height associated with $\mathfrak{p}$ over K as given in [Dis22, §1.3]. We consider the following hypothesis:

(**HGT**) $\langle z_{\mathrm{Heeg}}, z_{\mathrm{Heeg}} \rangle_K \neq 0$.

From Proposition 3.2 we can deduce the following:

Corollary 3.3. *Let $\mathcal{K}$ be a non-anticyclotomic $\mathbb{Z}_p$-extension of K. If (**HGT**) holds, then*

$$\left.\frac{\mathrm{d}\pi_K(L_{\mathfrak{p}}(\mathsf{A}))}{\mathrm{d}X_{\mathcal{K}}}\right|_{X_{\mathcal{K}}=0} \neq 0.$$

In particular, $\pi_{\mathcal{K}}(L_{\mathfrak{p}}(\mathsf{A})) \neq 0$.

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[1] Many authors, including [Dis17] refer to it as a p-adic L-function; but, to highlight the dependence on $\mathfrak{p} \mid p$, we refer to it as $\mathfrak{p}$-adic L-function.

4. Non-existence of non-trivial pseudonull submodules

In this section, we review a special case of a result of R. Greenberg [Gre16, Proposition 4.1.1] regarding sufficient conditions for $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ to admit no non-trivial pseudonull submodule.

Throughout this section, we keep the notation introduced previously. We further assume that

(**CYC**) $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ is Λ_{cyc}-torsion.

In view of [HO10, Lemma 2.6] (which is based on the ideas in [BH97]) we can conclude that (**CYC**) implies that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ is Λ_∞-torsion.

Let $\mathcal{T} = T_{\mathfrak{p}}(\mathsf{A}) \otimes \Lambda_\infty^\iota$, where ι is the involution on Λ_∞ sending a group-like element to its inverse and set $\mathcal{D} = \mathcal{T} \otimes_{\Lambda_\infty} \Lambda_\infty^\vee$. Throughout this section, we will be consistent with the notation of [Gre16, §2.1] as much as possible.

The condition $\mathbf{RFX}(\mathcal{D})$ asserts that $\mathcal{T}$ is a reflexive Λ_∞-module; in our setting this condition holds since it is free over Λ_∞.

In the context of our setup, the condition $\mathbf{LEO}(\mathcal{D})$ asserts that

$$\ker\left(H^2(K_\Sigma/K,\mathcal{D}) \longrightarrow \prod_{v\in\Sigma} H^2(K_v,\mathcal{D})\right)$$

is a cotorsion $\Lambda(G_K)$-module. Recall from [Gre06, Theorem 3] that there is an isomorphism of $\Lambda(G_K)$-modules $H^2(K_\Sigma/K,\mathcal{D}) \cong H^2(K_\Sigma/K_\infty,\mathsf{A}[\mathfrak{p}^\infty])$.

Since our hypotheses imply that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ is Λ_∞-torsion, we know (using arguments analogous to [OV03, Theorems 3.2]) that

$$\mathrm{rank}_{\Lambda_\infty} H^1(K_\Sigma/K_\infty,\mathsf{A}[\mathfrak{p}^\infty])^\vee - \mathrm{rank}_{\Lambda_\infty} H^2(K_\Sigma/K_\infty,\mathsf{A}[\mathfrak{p}^\infty])^\vee = r_2(K)\times r \tag{4.1}$$

where r is the $\mathcal{O}_{\mathfrak{p}}$-corank of $\mathsf{A}[\mathfrak{p}^\infty]$. We also know that

$$\mathrm{rank}_{\Lambda_\infty} \bigoplus_{\ell\in\Sigma} J_\ell(\mathsf{A}/K_\infty)^\vee = \mathrm{rank}_{\Lambda_\infty} \bigoplus_{\substack{w\mid\ell\\ \ell\in\Sigma}} H^1(K_{\infty,w},\mathsf{A})[\mathfrak{p}^\infty]^\vee = r_2(K)\times r, \tag{4.2}$$

by combining [CG96, Proposition 4.8] and [OV03, Theorem 4.1]. A standard argument with Poitou–Tate exact sequence implies that $H^2(K_S/K_\infty,\mathsf{A}[\mathfrak{p}^\infty])$ is co-torsion over Λ_∞ and $H^1(K_S/K_\infty,\mathsf{A}[\mathfrak{p}^\infty])$ is of corank $r_2(K)\times r$. In particular, $\mathbf{LEO}(\mathcal{D})$ holds.

The condition $\mathbf{CRK}(\mathcal{D},\mathcal{L})$ asserts that

$$\begin{aligned}\mathrm{corank}_{\Lambda_\infty} H^1(K_\Sigma/K_\infty,\mathsf{A}[\mathfrak{p}^\infty]) &= \mathrm{corank}_{\Lambda_\infty}\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty) + \mathrm{corank}_{\Lambda_\infty} \bigoplus_{\ell\in\Sigma} J_\ell(\mathsf{A}/K_\infty)\\ &= \mathrm{corank}_{\Lambda_\infty} \bigoplus_{\ell\in\Sigma} J_\ell(\mathsf{A}/K_\infty) \text{ since we assume (\textbf{CYC}).}\end{aligned}$$

The desired equality follows from (4.1) and (4.2).

We now consider the conditions $\mathbf{LOC}_v^{(i)}(\mathcal{D})$, $i=1,2$. Write $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D},\mu_{p^\infty})$. The conditions assert that for $v\in\Sigma(K)$, we have $(\mathcal{T}^*)^{G_{K_v}} = 0$ and $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is a reflexive Λ_∞-module, respectively. Since $p\neq 2$, we have $(\mathcal{T}^*)^{G_{K_v}} = 0$ when v is an archimedean prime. On the other hand, if v is a non-archimedean prime, it does not split completely in K_∞. It follows from [Gre10, Lemma 5.2.2] that $(\mathcal{T}^*)^{G_{K_v}} = 0$. This guarantees that condition $\mathbf{LOC}_v^{(1)}(\mathcal{D})$ holds for all $v\in\Sigma(K)$. Finally, as $\mathcal{T}^*$ is a free Λ_∞-module, $\mathbf{LOC}_v^{(2)}(\mathcal{D})$ both hold for all $v\in\Sigma(K)$.

We can now state the result due to Greenberg under the following condition:

(**TOR**) $\mathsf{A}(K)$ has no $\mathfrak{p}$-torsion.

Proposition 4.1. *If (**ORD**), (**TOR**) and (**CYC**) hold, then the Λ_∞-module $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ does not contain a non-trivial pseudonull submodule.*

Proof. We have verified that the hypotheses $\mathbf{RFX}(\mathcal{D})$, $\mathbf{LEO}(\mathcal{D})$, $\mathbf{CRK}(\mathcal{D},\mathcal{L})$, $\mathbf{LOC}_v^{(1)}(\mathcal{D})$, and $\mathbf{LOC}_v^{(2)}(\mathcal{D})$ hold for all $v \in \Sigma(K)$. Next, the condition $\mathcal{D}[\mathfrak{m}]$ admits no quotient isomorphic to μ_p for the action of G_K (assumption (b) in *loc. cit.*) is equivalent to $\mathsf{A}(K)[\mathfrak{p}] = 0$ via the Weil pairing (see the last paragraph on p. 248 of *op. cit.*). Therefore, the result is a direct consequence of [Gre16, Proposition 4.1.1]. □

5. Proof of Theorem A

5.1. **Preliminary results on Selmer groups.** Let A be a simple self-dual modular abelian variety of GL_2-type over a totally real field F and let K/F be a CM field, as before.

Under (**ORD**), (**CYC**) and (**TOR**), the structure theorem of finitely generated Λ_∞-modules (as given in [Bou65, Chapitre VII, §4, Théorème 5]) combined with Proposition 4.1 asserts the existence of the following short exact sequence

$$0 \longrightarrow \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee \longrightarrow \bigoplus_{i=1}^{m} \frac{\Lambda_\infty}{I_i} \longrightarrow N \longrightarrow 0, \tag{5.1}$$

where $I_1, \ldots, I_m$ are principal ideals of Λ_∞ and $\prod_{i=1}^m I_i = I = \mathrm{char}_{\Lambda_\infty}(\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee)$ is the characteristic ideal of $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ as a Λ_∞-module and N is a pseudonull Λ_∞-module.

Recall from the discussion in §3 the existence of a unique $\mathfrak{p}$-adic L-function $L_\mathfrak{p}(\mathsf{A}) \in F_\mathfrak{p} \otimes \Lambda_\infty$ associated with the abelian variety A, which is an analytic object. On the other hand, Theorem A is an assertion involving Selmer groups which are algebraic objects. To bridge this gap, we assume one inclusion of the Iwasawa main conjecture, i.e.,

(**h-IMC**) $L_\mathfrak{p}(\mathsf{A}) \in \mathrm{char}_{\Lambda_\infty}(\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee) = \prod_{i=1}^m I_i$.

Implicitly, (**h-IMC**) asserts that $L_\mathfrak{p}(\mathsf{A})$ lies inside Λ_∞.

In what follows, set $H_\mathcal{K} \simeq \mathrm{Gal}(K_\infty/\mathcal{K}) \simeq \mathbb{Z}_p^{\oplus d}$. When $\mathcal{K} = K_{\mathrm{cyc}}$, we abbreviate $H_{K_{\mathrm{cyc}}} = H_{\mathrm{cyc}}$.

Lemma 5.1. *Let $\mathcal{K}$ be a non-anticyclotomic $\mathbb{Z}_p$-extension of K. If* (**HGT**), (**ORD**), (**GHH**$^+$), (**TOR**), *and* (**h-IMC**) *hold, then* $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee_{H_\mathcal{K}}$ *is a finitely generated torsion* $\Lambda_\mathcal{K}$*-module.*

Proof. The short exact sequence (5.1) induces the exact sequence

$$H_1(H_\mathcal{K}, N) \longrightarrow \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee_{H_\mathcal{K}} \longrightarrow \bigoplus_{i=1}^{m} \frac{\Lambda_\mathcal{K}}{\pi_\mathcal{K}(I_i)} \longrightarrow H_0(H_\mathcal{K}, N) \longrightarrow 0.$$

The inclusion of (**h-IMC**) combined with Corollary 3.3 implies that $\pi_\mathcal{K}(I) \neq 0$. In particular, $\dfrac{\Lambda_\mathcal{K}}{\pi_\mathcal{K}(I_i)}$ is $\Lambda_\mathcal{K}$-torsion for $i = 1, \ldots, m$. This surjectivity of the last arrow in the exact sequence above implies that $H_0(H_\mathcal{K}, N)$ is $\Lambda_\mathcal{K}$-torsion. By [Lim15, Proposition 2.3], we conclude that $H_1(H_\mathcal{K}, N)$ is $\Lambda_\mathcal{K}$-torsion. Hence, we conclude that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee_{H_\mathcal{K}}$ is also a $\Lambda_\mathcal{K}$-torsion module. □

Proposition 5.2. *Write $\mathcal{K}/K$ to denote a $\mathbb{Z}_p$-extension, and let $H_\mathcal{K}$ denote the Galois group* $\mathrm{Gal}(K_\infty/\mathcal{K}) \simeq \mathbb{Z}_p^{\oplus d}$. *Suppose that* (**TOR**) *and* (**ORD**) *are satisfied. Then the restriction map*

$$\alpha : \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}) \longrightarrow \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^{H_\mathcal{K}}$$

is injective. When $\mathcal{K} = K_{\mathrm{cyc}}$, the cokernel of α is a torsion Λ_{cyc}-module. In particular, $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee_{H_{\mathrm{cyc}}}$ *is a torsion Λ_{cyc}-module if and only if* $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ *is a torsion* Λ_{cyc}*-module.*

We note that this result does not require any hypothesis on the reduction type at primes $v \mid p$.

Proof. We begin by recalling the fundamental diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}) & \longrightarrow & H^1(G_\Sigma(\mathcal{K}),\mathsf{A}[\mathfrak{p}^\infty]) & \xrightarrow{\ \pi\ } & \prod_{v\in\Sigma(\mathcal{K})} J_v(\mathsf{A}/\mathcal{K}) \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma=\prod\gamma_v} \\ 0 & \longrightarrow & \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^{H_\mathcal{K}} & \longrightarrow & H^1(G_\Sigma(K_\infty),\mathsf{A}[\mathfrak{p}^\infty])^{H_\mathcal{K}} & \longrightarrow & \prod_{v\in\Sigma(K_\infty)} J_v(\mathsf{A}/K_\infty)^{H_\mathcal{K}}, \end{array}$$

where the vertical maps are given by restriction. To prove the first assertion, we study the leftmost downward arrow and obtain the exact sequence

$$0\longrightarrow \ker(\alpha)\longrightarrow \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})\longrightarrow \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^{H_\mathcal{K}}.$$

Note that

$$\ker(\alpha)\hookrightarrow \ker(\beta)=H^1(H_\mathcal{K},\mathsf{A}(K_\infty)[\mathfrak{p}^\infty])$$

where the description of $\ker(\beta)$ comes from inflation-restriction. Since our assumptions imply that $\mathsf{A}(K)[\mathfrak{p}]=0$, it follows that $\mathsf{A}(K_\infty)[\mathfrak{p}]=0$ (see for example, [NSW08, I.6.13]). Thus,

$$\ker(\alpha)=\ker(\beta)=0.$$

The result is now immediate from our assumption that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee_{H_\mathcal{K}}$ is a $\Lambda_\mathcal{K}$-torsion module.

Next, we observe that the same argument as before yields

$$\mathrm{coker}(\beta)=H^2(H_\mathcal{K},\mathsf{A}(K_\infty)[\mathfrak{p}^\infty])=0.$$

To complete the proof, it suffices to show that

$$\mathrm{coker}(\alpha)=\ker(\gamma)=\prod_v \ker(\gamma_v)=\text{torsion }\Lambda_{\mathrm{cyc}}-\text{module}.$$

Note that $v\mid\mathfrak{p}$ is deeply ramified in the sense of Coates–Greenberg, imitating the proof of [CG96, Proposition 4.8] we obtain that (when $v\mid\mathfrak{p}$ has good ordinary reduction)

$$\ker(\gamma)=\prod\ker(\gamma_v)=\prod_{\substack{v\in\Sigma(\mathcal{K})\\ v\nmid\mathfrak{p}}} H^1(H_w,\mathsf{A}(K_{\infty,w})[\mathfrak{p}^\infty])\times\prod_{\substack{v\in\Sigma(\mathcal{K})\\ v\mid\mathfrak{p}}} H^1(H_w,\widetilde{\mathsf{A}}(k_w)[\mathfrak{p}^\infty]),$$

where $w=w_v$ denotes any place of K_∞ lying above v, H_w denotes the decomposition group of w inside $H_\mathcal{K}$, $\widetilde{\mathsf{A}}$ denotes the reduction of A modulo w, and k_w denotes the residue field of K_∞ at w.

For the remainder of the argument we assume that $\mathcal{K}=K_{\mathrm{cyc}}$. First, we consider the case that $v\nmid\mathfrak{p}$. In this case, the local map γ_v is simply the identity map: for any $v\in\Sigma(K_{\mathrm{cyc}})$ such that $v\nmid\mathfrak{p}$ and any place w of K_∞ lying above v, we have that $K_{\infty,w}=K_{\mathrm{cyc},v}$ is the unique $\mathbb{Z}_p$-extension of $K_{\mathfrak{p}'}$, where $\mathfrak{p}'$ is the place of K lying below v. In other words, $\ker(\gamma_v)=0$.

When $v\mid\mathfrak{p}$ and $\mathfrak{p}$ is a prime of good ordinary reduction, it suffices to know that $\widetilde{\mathsf{A}}(k_w)[\mathfrak{p}^\infty]$ itself is cotorsion (since it is of finite corank over $\mathcal{O}_\mathfrak{p}$). On the other hand, when $\mathfrak{p}$ is a prime of potentially ordinary reduction, $\ker(\gamma_v)=H^1(H_w,D)$ where $D=\mathsf{A}[\mathfrak{p}^\infty]/C$ where C is a formal group (over a base extension). In any case, the kernel lies inside $H^1(K_{\infty,v},\mathsf{A}[\mathfrak{p}^\infty])$. As a Λ_{cyc}-module, we note that $\mathsf{A}[\mathfrak{p}^\infty]$ is still cotorsion. So, $H^1(K_{\infty,v},\mathsf{A}[\mathfrak{p}^\infty])$ and hence $\ker(\gamma_v)$, is Λ_{cyc}-cotorsion. Here, we crucially use the fact that $v\mid\mathfrak{p}$ is finitely decomposed in the cyclotomic $\mathbb{Z}_p$-extension. ☐

5.2. **The (non)-growth of Selmer coranks.** We are now ready to conclude the proof of Theorem A, which is divided into two steps. The first step is to show that the Selmer corank is bounded in a non-anticyclotomic $\mathbb{Z}_p$-extension of K using the preliminary results from the previous sections. The second step is to show that the Selmer corank grows as predicted in an anticyclotomic $\mathbb{Z}_p$-extension of K. The main input will be results of Nekovář [Nek07, Nek08] on the Heegner point Euler system and the parity conjecture.

Theorem 5.3. *Let A be a simple modular self-dual abelian variety of GL_2-type over a totally real field F with trivial central character and K be a totally imaginary extension of F, satisfying* (**ORD**)*,* (**GHH**$^+$)*,* (**TOR**)*,* (**h-IMC**)*, and* (**HGT**)*. If $\mathcal{K}$ is a non-anticyclotomic $\mathbb{Z}_p$-extension of K and $\mathcal{K}_n$ is the n-th layer of the $\mathbb{Z}_p$-extension, then $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)^\vee$ is bounded as $n \to \infty$.*

Proof. In light of Theorem 2.2 and Corollary 2.3, it suffices to prove that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee$ is $\Lambda_{\mathcal{K}}$-torsion. By Proposition 5.2, there is a surjection

$$\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee_{H_{\mathcal{K}}} \twoheadrightarrow \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee.$$

Furthermore, it follows from Lemma 5.1 that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee_{H_{\mathcal{K}}}$ is $\Lambda_{\mathcal{K}}$-torsion. Hence, we deduce that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee$ is $\Lambda_{\mathcal{K}}$-torsion, as desired. □

Theorem 5.4. *Let A be a simple modular self-dual abelian variety of GL_2-type over a totally real field F with trivial central character and K a totally imaginary extension of F, satisfying* (**ORD**)*,* (**GHH**$^+$) *and* (**HGT**)*. Let $\mathcal{K}$ be a $\mathbb{Z}_p$-extension of K that lies inside K_{ac} and set $\mathcal{K}_n$ to denote the n-th layer of the $\mathbb{Z}_p$-extension $\mathcal{K}/K$. Then*

$$\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}} \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n) = p^n + O(1).$$

Proof. Under (**HGT**), z_{Heeg} is not a torsion element of $\mathsf{A}(K)$. In particular, the main theorem of [Nek07] implies that $Ш(\mathsf{A}/K)[\mathfrak{p}^\infty]$ is finite and the $\mathcal{O}_{\mathfrak{p}}$-module generated by z_{Heeg} inside $\mathsf{A}(K)$ is an $\mathcal{O}_{\mathfrak{p}}$-module of rank one. Therefore, $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)$ is of corank one over $\mathcal{O}_{\mathfrak{p}}$. As the restriction map

$$\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K) \to \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^{\Gamma_{\mathcal{K}}}$$

has finite kernel and cokernel by Theorem 2.2, it follows that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee$ is of rank one or zero over $\Lambda_{\mathcal{K}}$. However, since the root number of A over $\mathcal{K}_n$ is -1 by (**GHH**$^+$), it follows from [Nek08, Theorem 0.4] that the $\mathcal{O}_{\mathfrak{p}}$-corank of $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)$ is unbounded as $n \to \infty$. By Corollary 2.3 the $\Lambda_{\mathcal{K}}$-corank of $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})$ is one, and $\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}} \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n) = p^n + O(1)$, as desired. □

6. Proof of Theorem B

The main goal of this section is to prove Theorem B. We adopt the method utilized in [GHKL25]. In fact, we remove the non-anomalous hypothesis in *loc. cit.* and streamline the utilization of the control theorem. Instead of (**HGT**) in the previous section, we consider the following hypotheses:

(**S-C**) The Λ_∞-module $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee$ is a direct sum of cyclic Λ_∞ modules.

(**RK**) The order of vanishing of $\mathrm{char}_{\Lambda_{\mathrm{cyc}}} \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ at X_0 is at most one.

Theorem 6.1. *Let A be a simple modular self-dual abelian variety of GL_2-type over a totally real field F with trivial central character satisfying* (**ORD**)*. Let K/F be a totally imaginary extension and $\mathcal{K}/K$ be a $\mathbb{Z}_p$-extension. Suppose that* (**CYC**) *holds and either of following conditions holds:*

(*i*) *the order of vanishing of $\mathrm{char}_{\Lambda_{\mathrm{cyc}}} \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ at X_0 is zero;*

(*ii*) (**S-C**) *holds and $\mathcal{K}$ is a non-anticyclotomic $\mathbb{Z}_p$-extension of K.*

Then, $\mathrm{rank}_{\mathcal{O}_{\mathfrak{p}}} \mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)^\vee$ *is bounded as $n \to \infty$.*

Proof. In case (i), it follows from Theorem 2.2 that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)$ is finite; hence, $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee$ is $\Lambda_{\mathcal{K}}$-torsion. By Corollary 2.3, we conclude that the $\mathcal{O}_{\mathfrak{p}}$-rank of $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)^\vee$ is bounded as $n \to \infty$.

For the remainder of the proof, we consider case (ii). In particular, (**S-C**) holds, which means that the pseudonull module N in (5.1) is trivial. Let $f_i \in \Lambda_\infty$ be a generator of I_i. Let us write

$$f_i = \sum_{k=0}^{\infty} G_{i,k}(X_1, \ldots, X_d) X_0^k,$$

where $G_{i,j}(X_1,\ldots,X_d)\in\mathcal{O}_{\mathfrak{p}}[\![X_1,\ldots,X_d]\!]$. By Proposition 5.2, there is a surjection

$$\left(\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_\infty)^\vee\right)_{H_{\mathcal{K}}}\simeq\bigoplus_{i=1}^m\frac{\Lambda_{\mathcal{K}}}{\left(\pi_{\mathcal{K}}(f_i)\right)}\twoheadrightarrow\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee,\tag{6.1}$$

Thus, $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee$ is $\Lambda_{\mathcal{K}}$-torsion if $\pi_{\mathcal{K}}(f_i)\neq 0$ for all $i=1,\ldots,m$.

When $\mathcal{K}=K_{\mathrm{cyc}}$, Proposition 5.2 says that the kernel in (6.1) is Λ_{cyc}-torsion. Thus, combined with (**CYC**), we have

$$\pi_{\mathrm{cyc}}(f_i)=\sum_{k=0}^\infty G_{i,k}(0,\ldots,0)X_0^k\neq 0$$

for all $i\in\{1,\ldots,m\}$. In particular, there exists an integer $m(i)$ such that $G_{i,m(i)}(0,\ldots,0)\neq 0$, with $G_{i,k}(0,\ldots,0)=0$ for all $k<m(i)$.

Let $\mathcal{K}$ be a $\mathbb{Z}_p$-extension of K which is not an anticyclotomic extension. Recall that $\mathcal{K}$ corresponds to $(c_0:\cdots:c_d)\in\mathbb{P}^d(\mathbb{Z}_p)$ with $c_0\neq 0$. In particular,

$$\pi_{\mathcal{K}}(X_0)=(1+X_{\mathcal{K}})^{\frac{c_0}{c_j(\mathcal{K})}}-1=\frac{c_0}{c_j(\mathcal{K})}X_{\mathcal{K}}+O(X_{\mathcal{K}}^2).$$

Therefore,

$$\begin{aligned}\pi_{\mathcal{K}}(f_i)&=\sum_{k\geq m(i)}G_{i,k}\left((1+X_{\mathcal{K}})^{\frac{c_1}{c_j(\mathcal{K})}}-1,\ldots,(1+X_{\mathcal{K}})^{\frac{c_d}{c_j(\mathcal{K})}}-1\right)\pi_{\mathcal{K}}(X_0)^k\\&=G_{i,m(i)}\left((1+X_{\mathcal{K}})^{\frac{c_1}{c_j(\mathcal{K})}}-1,\ldots,(1+X_{\mathcal{K}})^{\frac{c_d}{c_j(\mathcal{K})}}-1\right)\left(\frac{c_0}{c_j(\mathcal{K})}X_{\mathcal{K}}\right)^{m(i)}+O(X_{\mathcal{K}}^{m(i)+1})\\&=G_{i,m(i)}(0,\ldots,0)\left(\frac{c_0}{c_j(\mathcal{K})}X_{\mathcal{K}}\right)^{m(i)}+O(X_{\mathcal{K}}^{m(i)+1})\neq 0.\end{aligned}$$

Hence, we deduce that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K})^\vee$ is $\Lambda_{\mathcal{K}}$-torsion. Thus, Corollary 2.3 tells us that the $\mathcal{O}_{\mathfrak{p}}$-rank of $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)^\vee$ is bounded as $n\to\infty$. □

Theorem 6.2. *Let* A *be as in Theorem 6.1. Suppose that* (**ORD**), (**GHH$^+$**), (**CYC**) *and* (**RK**) *hold. If* $\mathcal{K}$ *is a* $\mathbb{Z}_p$*-extension of* K *that lies inside* K_{ac}*, then*

$$\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}}\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/\mathcal{K}_n)=p^n+O(1).$$

Proof. As in the proof of Theorem 5.4, it suffices to show that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)^\vee$ is of corank one over $\mathcal{O}_{\mathfrak{p}}$. Indeed, under (**GHH$^+$**), the corank of $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)^\vee$ is non-zero by [Nek07, Theorem 0.4]. Therefore, combined with (**RK**), the order of vanishing of $\mathrm{char}_{\Lambda_{\mathrm{cyc}}}\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K_{\mathrm{cyc}})^\vee$ at X_0 is exactly one. Thus, Theorem 2.2 implies that $\mathrm{Sel}_{\mathfrak{p}^\infty}(\mathsf{A}/K)^\vee$ is of corank one over $\mathcal{O}_{\mathfrak{p}}$, as desired. □

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