GUEST LECTURE: MATH CAMP (AUGUST 16 2019)

GOAL: To show that answers to simple questions about "simple functions" (eg. the Möbius function) are related to quite deep facts about prime numbers, in particular the Prime Number Theorem and the Riemann Hypothesis.

Definition. *An arithmetic function $f(n) : \mathbb{N} \to \mathbb{C}$ is called* ***multiplicative*** *if for any relatively prime $n, m \in \mathbb{N}$,*

$$f(mn) = f(m)f(n).$$

Example. Let $n \in \mathbb{N}$. Define the following functions $\tau, \sigma, \mu, \phi, \Pi : \mathbb{N} \to \mathbb{N}$ as follows

- $\tau(n) =$ the number of all natural divisors of n.
- $\sigma(n) =$ the sum of all natural divisors of n.
- $\Pi(n) =$ the product of all natural numbers of n.
- $\phi(n) =$ the Euler totient function
- $\mu(n) =$ the Möbius function.

Exercise 1. *I leave it as an exercise to check that if $n = p_1^{e_1} \cdots p_r^{e_r}$ with $e_i \geq 1$; then*

$$\tau(n) = \prod_{i=1}^{r}(e_i + 1), \qquad \sigma(n) = \prod_{i=1}^{r} \frac{p_i^{e_i+1} - 1}{p_i - 1}, \qquad \Pi(n) = n^{\frac{1}{2}\tau(n)}.$$

It is now easy to see that τ, σ are multiplicative but Π is not.

Definition. *The Möbius function $\mu : \mathbb{N} \to \mathbb{C}$ is defined by*

$$\mu(n) = \begin{cases} 1; & \text{if } n = 1 \\ (-1)^r; & \text{if } n = p_1 \cdots p_r,\ p_i \text{ are all distinct.} \\ 0; & \text{if } n \text{ is not square-free.} \end{cases}$$

Exercise 2. *For the advanced members of the audience:*

$$\left| \sum_{k=1}^{n} \mu(k) \right| \leq n^{\frac{1}{2}+\epsilon} \quad \text{for } \epsilon > 0.$$

One of the main reasons for this function to be relevant in the theory of multiplicative functions is the following formula.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases} \tag{1}$$

Write $n = p_1^{a_1} \cdots p_r^{a_r}$ as before. If $d \mid n$, $d = p_1^{b_1} \cdots p_r^{b_r}$ with $0 \leq b_i \leq a_i$. If $b_i \geq 2$, then $\mu(d) = 0$.

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$$\begin{aligned}\sum_{d|n}\mu(d) &= \sum_{b_1,\cdots,b_r}\mu\left(p_1^{b_1}\cdots p_r^{b_r}\right)\\ &= 1-\binom{r}{1}+\binom{r}{2}-\cdots+(-1)^r\binom{r}{r}\\ &= (1-1)^r\\ &= 0\end{aligned}$$

This gives rise to next two theorems, which we state without proof.

Theorem. ***Möbius Inversion Formula:*** *Let f,g be two arithmetic functions such that for all non-negative integers, n we have*

$$g(n)=\sum_{d|n}f(d);$$

then

$$f(n)=\sum_{d|n}\mu(d)g\left(\frac{n}{d}\right).$$

Theorem. ***Multiplicative Möbius Inversion Formula:*** *Let f,g be two arithmetic functions such that for all negative integers, n we have*

$$g(n)=\prod_{d|n}f(d);$$

then

$$f(n)=\prod_{d|n}g\left(\frac{n}{d}\right)^{\mu(d)}=\prod_{d|n}g(d)^{\mu\left(\frac{n}{d}\right)}.$$

Definition. ***Euler totient function:*** *This function, denoted by $\phi(n)$ is the number of positive integers less than n that are coprime to n.*

Exercise 3. *Show that $\sum_{d|n}\phi(d)=n$.*

Möbius Inversion formula immediately gives

$$\phi(n)=\sum_{d|n}\mu(d)\frac{n}{d}$$

$$\frac{\phi(n)}{n}=\sum_{d|n}\frac{\mu(d)}{d}.$$

Interesting Formulas involving the Möbius Function

A common strategy to prove facts about multiplicative functions is to first restrict attention to their values on prime powers. That is, if two multiplicative functions agree on prime powers, they must agree everywhere.

Example. Let $\omega(n)$ denote the number of distinct prime factors of n. Then

$$\sum_{d|n}\mu(d)^2=2^{\omega(n)}.$$

Note: $2^{\omega(n)}$ is multiplicative as $\omega(ab)=\omega(a)+\omega(b)$ if $\gcd(a,b)=1$. OK, so both sides are multiplicative. For $n=p^k$, LHS $=1+1$ and the RHS $=2$.

Exercise 4. *Prove that*

$$\sum_{d|n} \frac{\mu(d)^2}{\phi(d)} = \frac{n}{\phi(n)}.$$

Relating the Möbius Function to the Riemann ζ Function

Definition. *A **generating function** $f(x)$ is a formal power series*

$$f(x) = \sum_{n \geq \infty} a_n x^n$$

whose coefficients give the sequence $\{a_0, a_1, \dots,\}$

Example.

$$\begin{aligned} f(x) &= \frac{x}{1 - x - x^2} \\ &= \sum_{n \geq 0} F_n x^n \\ &= x + x^2 + 2x^3 + 3x^4 + \dots \end{aligned}$$

for the Fibonacci numbers F_n.

The **Riemann ζ function** is the following sum

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(\frac{1}{1 - p^{-s}} \right)$$

where $s \in \mathbb{C}$ with $\Re(s) > 1$.

When $\Re(s) \leq 1$, one can has the following equation that relates the values of the Riemann zeta function at the points s and $1 - s$, in particular it relates the even positive integers with odd negative integers.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s) \zeta(1 - s).$$

The zeroes of the sine function force that at $s = -2n$, $\zeta(s) = 0$. When s is an even positive integer, the product $\sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s)$ is non-zero.

The famous **Riemann Hypothesis** is a statement about the non-trivial zeroes of this function.

Now, we will relate the Riemann ζ function to the Möbius function.

$$\begin{aligned} \frac{1}{\zeta(s)} &= \prod_{k \geq 1} \left(1 - \frac{1}{p_k^s} \right) \\ &= \left(1 - \frac{1}{p_1^s} \right) \left(1 - \frac{1}{p_2^s} \right) \cdots \\ &= 1 - \left(\frac{1}{p_1^s} + \frac{1}{p_2^s} + \frac{1}{p_3^s} + \cdots \right) + \left(\frac{1}{p_1^s p_2^s} + \frac{1}{p_1^s p_3^s} + \cdots + \frac{1}{p_2^s p_3^s} + \frac{1}{p_2^s p_4^s} + \cdots \right) - \cdots \\ &= 1 - \sum_{i > 0} \frac{1}{p_i^s} + \sum_{j > i > 0} \frac{1}{p_i^s p_j^s} - \cdots \\ &= \sum_{n \geq 1} \frac{\mu(n)}{n^s} \end{aligned}$$

So,we shown that Möbius function has "generating functions"

$$\sum_{n\geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

for $\Re(s) > 1$.

This is not a "formal power series". This is what we call a **Dirichlet Series**.

(1) Start with all numbers (if you want, you may think of it as all numbers in some large interval). They are $\frac{1}{1}$ of the total.
(2) Remove all even numbers. These are $\frac{1}{2}$ of the total.
(3) Remove all numbers divisible by 3 from the total.
(4) But numbers divisible by 6 (these are $\frac{1}{6}$ of the total) have been thrown out twice, so we put them back once. So on...

Goal is to remove every number exactly one time... then we end up with 0

Writing it down mathematically

$$\frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} + ... = \sum_{n\geq 1} \frac{\mu(n)}{n} = 0.$$

This is equivalent to the Prime Number Theorem, ie $\pi(x) \sim \frac{x}{\log x}$. Here $\pi(x)$ is the prime counting function.