1. Introduction

My primary research is on the Iwasawa theory of elliptic curves. I use tools from Galois cohomology, module theory, algebraic and analytic number theory, and arithmetic statistics to answer questions on the structure and growth of Selmer groups, fine Selmer groups, and class groups in infinite (and finite) field extensions.

Organization: In §2, basic definitions and notations are introduced. In §3, I discuss some of my results on growth questions of *fine* Selmer groups. These results indicate that the fine Selmer group "interpolates" between the class group and the Selmer group. In §4, I explain some of my results which lie at the intersection of arithmetic statistics and Iwasawa theory. If you are interested in knowing about my ongoing/ future projects, please send me an email at `dkundu@math.ubc.ca`.

2. Basic Definitions and Notations

Let p be a fixed prime. Consider the *cyclotomic* $\mathbb{Z}_p$-extension of $\mathbb{Q}$, denoted by $\mathbb{Q}_{\mathrm{cyc}}$. Set $\Gamma := \mathrm{Gal}\left(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}\right) \simeq \mathbb{Z}_p$. The *Iwasawa algebra* $\Lambda = \Lambda(\Gamma)$ is the completed group algebra $\mathbb{Z}_p[\![\Gamma]\!] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. Fix a topological generator γ of Γ; there is the following isomorphism of rings

$$\begin{aligned} \Lambda &\xrightarrow{\sim} \mathbb{Z}_p[\![T]\!] \\ \gamma &\mapsto 1+T. \end{aligned}$$

Let M be a cofinitely generated cotorsion Λ-module. The *Structure Theorem of* Λ*-modules* asserts that the Pontryagin dual of M, denoted by $M^\vee$, is pseudo-isomorphic to a finite direct sum of cyclic Λ-modules. In other words, there is a map of Λ-modules

$$M^\vee \longrightarrow \left(\bigoplus_{i=1}^{s} \Lambda/(p^{m_i})\right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(h_j(T))\right)$$

with finite kernel and cokernel. Here, $m_i > 0$ and $h_j(T)$ is a distinguished polynomial (i.e., a monic polynomial with non-leading coefficients divisible by p). The *characteristic ideal* of $M^\vee$ is (up to a unit) generated by the *characteristic element*,

$$f_M^{(p)}(T) := p^{\sum_i m_i} \prod_j h_j(T).$$

The μ-invariant of M is defined as the power of p in $f_M^{(p)}(T)$. More precisely,

$$\mu(M) = \mu_p(M) := \begin{cases} 0 & \text{if } s = 0 \\ \sum_{i=1}^{s} m_i & \text{if } s > 0. \end{cases}$$

The λ-invariant of M is the degree of the characteristic element, i.e.

$$\lambda(M) = \lambda_p(M) := \sum_{j=1}^{t} \deg h_j(T).$$

3. Research Focus I: Fine Selmer Groups

The notion of *fine Selmer group* was formally introduced by J. Coates and R. Sujatha in [4] even though it had been studied by K. Rubin [49] and B. Perrin-Riou [45, 46], under various guises in the late 80's and early 90's. This is a subgroup of the classical Selmer group obtained by imposing stronger vanishing conditions at primes above p (the precise definition is reviewed in §3.1 below). A deep result of K. Kato shows that the fine Selmer group of an elliptic curve over $\mathbb{Q}_{\mathrm{cyc}}$ is always Λ-cotorsion, regardless of whether the elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ is ordinary at p or not, a fact that is *not* true for classical Selmer groups. The fine Selmer group is a fundamental object in the study of Iwasawa theory and plays a crucial role in the reformulation of the Iwasawa Main Conjecture for elliptic curves without p-adic L-functions (see [16, Conjecture 12.10] and [56, Conjecture 7]).

3.1. **Definition of fine Selmer groups.** Suppose F is a number field. Throughout, $\mathsf{E}_{/F}$ is a fixed elliptic curve. Fix a finite set S of primes of F containing p, the primes dividing the conductor of E, as well as the archimedean primes. Denote by F_S, the maximal algebraic extension of F unramified outside S. For every (possibly infinite) extension L/F contained in F_S, write $G_S(L) = \mathrm{Gal}(F_S/L)$. Write $S(L)$ for the set of primes of L above S. If L is a finite extension of F and w is a place of L, write L_w for its completion at w; when L/F is infinite, it is the union of completions of all finite sub-extensions of L.

Definition. Let L/F be an algebraic extension. The *p-primary fine Selmer group of* E over L is defined as

$$\mathrm{Sel}_0(\mathsf{E}/L) = \ker\left(H^1\left(G_S(L), \mathsf{E}[p^\infty]\right) \to \bigoplus_{v\in S} H^1\left(L_v, \mathsf{E}[p^\infty]\right)\right).$$

Similarly, the *p-fine Selmer group of* E over L is defined as

$$\mathrm{Sel}_0(\mathsf{E}[p]/L) = \ker\left(H^1\left(G_S(L), \mathsf{E}[p]\right) \to \bigoplus_{v\in S} H^1\left(L_v, \mathsf{E}[p]\right)\right).$$

Equivalently, the p-primary fine Selmer group is a subgroup of the classical Selmer group with additional vanishing conditions at $v \mid p$. More precisely,

$$0 \to \mathrm{Sel}_0\left(\mathsf{E}/F\right) \to \mathrm{Sel}\left(\mathsf{E}/F\right) \to \bigoplus_{v\mid p} \mathsf{E}(F_v)\otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

It is easy to observe that if F_∞/F is an infinite extension,

$$\mathrm{Sel}_0\left(\mathsf{E}/F_\infty\right) = \varinjlim_L \mathrm{Sel}_0\left(\mathsf{E}/L\right), \quad \mathrm{Sel}_0\left(\mathsf{E}[p]/F_\infty\right) = \varinjlim_L \mathrm{Sel}_0\left(\mathsf{E}[p]/L\right),$$

where the inductive limits are taken with respect to the restriction maps and L runs over all finite extensions of F contained in F_∞.

3.2. $\mu = 0$ **Conjecture for fine Selmer groups.** In [42], B. Mazur initiated the study of Iwasawa theory of classical Selmer groups of elliptic curves. Even over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$, when the Selmer group is Λ-cotorsion (i.e., at an odd prime p of good *ordinary* reduction), he constructed examples of elliptic curves where the μ-invariant of the Selmer group is *positive*. Thus, providing evidence that class groups and Selmer groups often differ in growth behaviour in infinite extensions. On the other hand, when $\mathsf{E}_{/F}$ is an elliptic curve with good ordinary reduction at p and the residual Galois representation $\overline{\rho} : \mathrm{Gal}\left(\overline{F}/F\right) \to \mathrm{GL}_2(\mathbb{F}_p)$ is *irreducible*, R. Greenberg has conjectured that the associated μ-invariant of the Selmer group over F_{cyc} is trivial (see [9, Conjecture 1.11]).

Motivated by the *classical Iwasawa $\mu = 0$ conjecture* for the *cyclotomic* $\mathbb{Z}_p$-extension and the observation that the growth behaviour of the fine Selmer group parallels that of the class group [4, Lemma 3.8], Coates–Sujatha formulated the following conjecture . Henceforth, this will be referred to as *Conjecture A*.

Conjecture A. Let p be an odd prime and $\mathsf{E}_{/F}$ be an elliptic curve. Over F_{cyc}/F, the fine Selmer group $\mathrm{Sel}_0(\mathsf{E}/F_{\mathrm{cyc}})$ is a cofinitely generated $\mathbb{Z}_p$-module. Equivalently, $\mathrm{Sel}_0(\mathsf{E}/F_{\mathrm{cyc}})$ is Λ-cotorsion and the associated μ-invariant, denoted by $\mu_{\mathrm{fine}}(\mathsf{E}/F_{\mathrm{cyc}})$, is 0.

Much of my research has been driven by trying to understand this conjecture. Even though we are far from proving this conjecture in full generality, I have made some modest contributions towards this (see Corollary 4.3). In my doctoral thesis, I provided a large class of examples where the conjecture is true (see [25, 32]). More precisely,

Theorem 3.1. *Let $\mathsf{E}_{/F}$ be the base-change of a rational elliptic curve $\mathsf{E}_{/\mathbb{Q}}$. Suppose that it has rank 0 over F and that the Shafarevich–Tate group of $\mathsf{E}_{/F}$ is finite. If E has CM by an order if an imaginary quadratic field K, assume further that the Galois closure of F, denoted by F^c, contains K. Then the Selmer group $\mathrm{Sel}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ is trivial for a set of prime numbers of density at least $\frac{1}{[F^c:\mathbb{Q}]}$. In particular, Conjecture A holds for $\mathsf{E}_{/F}$ at all such primes.*

The key difficulty in extending this result to elliptic curves defined over F is that we rely on [43] to show that anomalous primes have density 0. Since these results are proven for normalized weight 2 eigencuspforms, we need to invoke the Modularity Theorem. This has now been extended to higher rank elliptic curves and primes of good supersingular reduction (see §4).

3.3. **Close relationship with class groups.** In [40], M. F. Lim and V. K. Murty showed for the first time that the class groups of number fields and fine Selmer groups are closely related in some finite extensions of number fields and in $\mathbb{Z}_p$-extensions (other than the cyclotomic one) where primes are finitely decomposed.

3.3.1. **Arbitrarily large μ-invariant.** In my doctoral thesis, I explored similar questions in $\mathbb{Z}/p\mathbb{Z}$-extensions, non p-adic analytic extensions, and other p-adic Lie extensions. A particular question I was interested in studying was whether the growth properties of fine Selmer groups mimics that of class groups even in those p-adic Lie extensions where primes *are not finitely decomposed*. K. Iwasawa had shown in [15] the existence of $\mathbb{Z}_p$-extensions of certain number fields where the classical μ-invariant can be made *arbitrarily large*. In [23], I proved that an analogous result holds for fine Selmer groups by comparing p-ranks of fine Selmer groups and class groups in towers of number fields, and observing that the μ-invariant is closely related to the p-rank. More precisely,

Theorem 3.2. *Let $F = \mathbb{Q}(\zeta_p)$ be the cyclotomic field of p-th roots of unity for $p > 2$. Let $\mathsf{E}_{/F}$ be an elliptic curve such that $\mathsf{E}(F)[p] \neq 0$. Given an integer $N \geq 1$, there exists a cyclic Galois extension L/F of degree p and a* non-cyclotomic *$\mathbb{Z}_p$-extension L_∞/L such that $\mu_{\text{fine}}(\mathsf{E}/L_\infty) \geq N$.*

In [24], I *developed a strategy* to show that the (generalized) μ-invariant of fine Selmer groups can be arbitrarily large in extensions where the (generalized) μ-invariant associated to the class group is arbitrarily large. Using results of [11], I provided *explicit examples* of commutative and non-commutative p-adic Lie extensions with arbitrarily large (generalized) μ-invariant of fine Selmer groups. A striking feature in *all* these examples is that there are infinitely many primes which are infinitely decomposed in these extensions. This raises the following question:

Question 3.3. Should one expect that for any $\mathbb{Z}_p$-extension, where primes are finitely decomposed the classical Iwasawa μ-invariant (i.e., associated to class groups) is 0? More generally, if F_∞/F is a (uniform) pro-p p-adic Lie extension where primes are finitely decomposed, is the (generalized) μ-invariant trivial?

3.3.2. **Anti-cyclotomic $\mathbb{Z}_p$-extension.** In [41, Conjecture B], A. Matar extended Conjecture A to the *anti-cyclotomic* $\mathbb{Z}_p$-extension, K_{ac}, of an imaginary quadratic field, K. He provided computational evidence for the same when the mod-p representation of E is *irreducible*. In contrast, when the residual representation in *reducible*, we proved the following result in [33] which again underlines the relationship between class groups and fine Selmer groups.

Theorem 3.4. *Let E be an elliptic curve defined over an imaginary quadratic field K. Assume that*

(*i*) $\mathsf{E}(K)[p] \neq 0$ and
(*ii*) *the Heegner hypothesis is satisfied.*

Then the classical (anti-cyclotomic) Iwasawa μ-invariant, $\mu(K_{\text{ac}}/K) = 0$ if and only if $\mathrm{Sel}_0(\mathsf{E}/K_{\text{ac}})$ is a cotorsion Λ-module with $\mu_{\text{fine}}(\mathsf{E}/K_{\text{ac}}) = 0$.

3.3.3. $p \neq q$ **Iwasawa theory.** In [54, 55], L. C. Washington proved that for distinct primes p and q, the p-part of the class number stabilizes in the *cyclotomic* $\mathbb{Z}_q$-extension of an abelian number field. These results were extended by J. Lamplugh in [37] to other $\mathbb{Z}_q$-extensions where primes are finitely decomposed. More precisely, if p, q are distinct primes ≥ 5 that split in an imaginary quadratic field K of class number 1 and F/K is a prime-to-p abelian extension which is unramified at p, then the p-class group stabilizes in the $\mathbb{Z}_q$-extension of F which is unramified outside precisely one of the primes above q. Using a theorem of H. Hida on the non-vanishing modulo p of algebraic L-functions, we have extended these results to a class of *anti-cyclotomic* $\mathbb{Z}_p$-extensions in joint work with A. Lei [27].

Theorem 3.5. *Let K be an imaginary quadratic field of class number 1. Let p and q be distinct primes (≥ 5) which split in K. Let $\mathfrak{g}$ be a fixed ideal of $\mathcal{O}_K$ coprime to pq such that $\mathfrak{g}$ is a product of split primes. Let $F = \mathscr{R}(\mathfrak{g}q)$ be a prime-to-p extension of K and $\mathscr{R}(\mathfrak{g}q^\infty)^{\text{ac}}/F$ be the anti-cyclotomic $\mathbb{Z}_q$-extension. Then, there exists an integer N such that for all $n \geq N$,*

$$\operatorname{ord}_p\left(h\left(F_n\right)\right) = \operatorname{ord}_p\left(h\left(F_N\right)\right),$$

where F_n is the n-th layer of the anti-cyclotomic $\mathbb{Z}_q$-extension.

Main idea of the proof: Proving the theorem involves the following main steps:

Step 1: A special case of a theorem of Hida [14] guarantees the non-vanishing modulo p of algebraic L-functions.

Step 2: Using a result of Lamplugh, show that certain p-primary Galois modules stabilize in the said anti-cyclotomic $\mathbb{Z}_q$-extension. The theorem follows by an application of the Iwasawa Main Conjecture, which is known in this setting by the work of K. Rubin [48].

Even in this $p \neq q$ setting, it is possible to relate the growth of the p-part of the class group to the p-part of fine Selmer group of a fixed elliptic curve E over a $\mathbb{Z}_q$-tower. More precisely,

Theorem 3.6. *Let p and q be distinct odd primes. Let F be any number field and let $\mathsf{E}_{/F}$ be an elliptic curve such that $\mathsf{E}(F)[p] \neq 0$. Let F_∞/F be* any *$\mathbb{Z}_q$-extension such that the primes above q and the primes of bad reduction of E are* finitely decomposed. *If there exists $N \gg 0$ such that $\operatorname{ord}_p\left(h\left(F_n\right)\right) = \operatorname{ord}_p\left(h\left(F_N\right)\right)$ for all $n \geq N$, then $\operatorname{Sel}_0(\mathsf{E}/F_n) = \operatorname{Sel}_0(\mathsf{E}/F_N)$.*

In particular, the above theorem applies to the setting studied by Washington [54, 55].

A related question for exploration by early graduate students: I am hopeful that the strategy explained in Section 3.3.1 can be combined with the results in this section. More precisely, I think it should be possible to show that in $\mathbb{Z}_q$-extensions where primes of bad reduction of E split completely, the p-primary fine Selmer group has unbounded (but predictable) growth.

The result I describe below is not related to the theory of fine Selmer groups. However, the questions arose naturally while working on [27]. It is a classical problem to study the divisibility of the algebraic part of (Hecke) L-values by a given prime p as one varies the (Hecke) characters of q-power conductor. W. Sinnott introduced the idea of relating non-vanishing of such L-values modulo p to Zariski density (modulo p) of special points of the algebraic variety underlying the L-values. Related questions have been studied by A. Burungale, T. Finis, Hida, and M. L. Hsieh on anti-cyclotomic characters. In [28], which is joint work with Lei, we studied a generalization of the aforementioned results to Hecke characters (not necessarily anti-cyclotomic) of q-power conductor over an imaginary quadratic field.

Theorem 3.7. *Let K be an imaginary quadratic field over which p and q are both split and H be its Hilbert class field. Suppose that q does not divide the class number of K and that both the prime ideals above q are principal in K. Let $\mathsf{E}_{/H}$ be an elliptic curve with CM by $\mathcal{O}_K$ which has good reduction at primes above p and q. Let φ be the Hecke character over K attached to E of conductor $\mathfrak{f}$. Let $\mathfrak{g}$ denote a principal integral ideal of K that is divisible by $\mathfrak{f}$ and coprime to pq. Let $F = H(\mathsf{E}_{\mathfrak{g}q})$ and $F_\infty = \bigcup_{n\geq 1} H(\mathsf{E}_{\mathfrak{g}q^n})$ be the $\mathbb{Z}_q^2$-extension over F. Let π be the uniformizer of the local field F_v where $v \mid p$. Let ρ be a character of* $\operatorname{Gal}(H/K)$ *satisfying a technical hypothesis. Then, for a Zariski dense set of finite-order characters κ of* $\operatorname{Gal}\left(F_\infty/F\right)$,

$$\operatorname{ord}_\pi\left(L^{\mathrm{alg}}\left(\overline{\kappa\rho\varphi^k}\right)\right) = 0$$

for $k = 1, 2, \ldots, p-1$.

Main idea of the proof: The proof consists of the following main ingredients:

Step 1: Establish a theory of Gamma transform of "elliptic function measures" on $\mathbb{Z}_q^2$, which are measures that arise from a rational function on an elliptic curve.

Step 2: Show that the π-adic valuations of the aforementioned Gamma transforms have the same p-adic valuation for almost all finite characters on $\mathbb{Z}_q^2$.

Step 3: Show that by choosing an appropriate elliptic function measure arising from a rational function on a CM elliptic curve, the Gamma transforms of this measure is related to the special values of L-series of interest. The construction of this elliptic function measure is significantly more involved than the analogous step in the work of Lamplugh (since he assumes that K has class number 1). This step crucially uses the work of E. de Shalit.

Step 4: Show that the π-adic valuation discussed in Step 2 is zero.

3.4. **Close relationship with classical Selmer groups.** Since the fine Selmer group is a subgroup of the classical Selmer group, unsurprisingly these two arithmetic objects often show some similarity in their growth behaviour. One instance where this has been observed, in a limited scope, is in the formula of the cyclotomic λ-invariants via a Kida-type formula, see [26].

3.4.1. **Control Theorems.** In [42], Mazur conjectured that the *classical* Selmer group $\operatorname{Sel}(\mathsf{E}/F_{\mathrm{cyc}})$ is Λ-cotorsion, and also provided the first theoretical evidence towards the same. Using the *Control Theorem*, he verified the conjecture when $\operatorname{Sel}(\mathsf{E}/F)$ is finite. This condition is satisfied precisely when the Shafarevich–Tate group over F is finite and the elliptic curve $\mathsf{E}_{/F}$ has Mordell–Weil rank 0. Till date, this conjecture is known only when E is an elliptic curve over $\mathbb{Q}$ and F is an abelian extension of $\mathbb{Q}$; see [16].

Via the *Iwasawa Main Conjecture*, the Selmer group $\operatorname{Sel}(\mathsf{E}/F_{\mathrm{cyc}})$ can be related to a p-adic L-function. Therefore, Mazur's Control Theorem provides a channel to extract information on $\operatorname{Sel}(\mathsf{E}/F)$ from the main conjecture thereby providing an invaluable approach towards studying the Birch and Swinnerton-Dyer Conjecture (see [16, 48, 52]). The Control Theorem connects the Selmer groups at the finite layers with that over the infinite tower, allowing one to deduce properties of this arithmetic object over the infinite tower from those at the finite layers, and vice versa.

In joint work with M. F. Lim [31], we proved Control Theorems for fine Selmer groups. More precisely,

(a) established estimates on the $\mathbb{Z}_p$-coranks of the kernel and cokernel of the restriction maps

$$r_{F_\infty/F'} : \mathrm{Sel}_0(\mathsf{E}/F') \longrightarrow \mathrm{Sel}_0(\mathsf{E}/F_\infty)^{\mathrm{Gal}\left(F_\infty/F'\right)}$$

for a p-adic Lie extension F_∞/F with intermediate sub-fields F'/F.

(b) showed how the module theoretic structure of $\mathrm{Sel}_0(\mathsf{E}/F_\infty)$ determines the growth of $\mathbb{Z}_p$-coranks of $\mathrm{Sel}_0(\mathsf{E}/F')$ in intermediate sub-fields F'.

(c) obtained sharper results by specializing to three cases of p-adic Lie extensions: $\mathbb{Z}_p^d$-extensions, multi-false-Tate extensions, and the trivializing extension obtained by adjoining to F all the p-power division points of the elliptic curve, E. In each of these cases, it is possible to show (under appropriate assumptions) that the kernel and cokernel of the restriction map are *finite*, and also establish growth estimates for their orders.

Our results though stated only for elliptic curves go through for abelian varieties rather formally. This has been written down in [19] and has been used extensively in proving new asymptotic formulas for the growth of ideal class groups and fine Selmer groups in multi-$\mathbb{Z}_p$-extensions. More recently, the ideas from our paper have been extended to modular forms in [20] to study generalized Iwasawa invariants.

3.4.2. **Fine Selmer groups and duality.** In [8], Greenberg established a criteria for when two finitely generated Λ-modules are pseudo-isomorphic. This result has been used to show that the Selmer groups of ordinary (resp. non-ordinary) representations satisfy a functional equation in [8] (resp. [17, 1, 38]). A key ingredient in all these works is that the local Selmer conditions at p are *exact annihilator of each other*. Unfortunately, this is *not true* in the case of fine Selmer groups, since the local conditions at p are trivial. In joint work with J. Hatley, A. Lei, and J. Ray [12], we investigated the link between fine Selmer groups of weight k modular forms and its dual. More precisely, let f be a weight $k(\geq 2)$ modular form and $\overline{f}$ be the conjugate modular form. Then,

(a) for an integer i, several control theorems were proven for the fine Selmer groups of $f(i)$ and $\overline{f}(k-i)$.

(b) via global duality and global Euler characteristic formulae, it could be shown that the criteria established by Greenberg can be reinterpreted in terms of growth conditions on the localization maps.

(c) under hypotheses which could be verified computationally, using the control theorems it was shown that the growth conditions on certain localization maps suffice to study the relation between the fine Selmer groups of $f(i)$ and $\overline{f}(k-i)$ over $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$. In particular, if certain naturally arising growth conditions of localization maps are satisfied, the μ-invariants of $\mathrm{Sel}_0(f(i)/\mathbb{Q}_{\mathrm{cyc}})$ and $\mathrm{Sel}_0(\overline{f}(k-i)/\mathbb{Q}_{\mathrm{cyc}})$ are equal.

4. Research Focus II: Arithmetic Statistics and Iwasawa Theory

In a series of articles (with collaborators), I am exploring questions at the intersection of arithmetic statistics and Iwasawa theory. The main goal is to understand the variation of the Iwasawa invariants as the triple (E, F, p) varies such that $\mathsf{E}_{/F}$ has good reduction at p. More precisely, I focus on studying the following interrelated problems.

Question 4.1. (i) For a fixed elliptic curve $\mathsf{E}_{/F}$, how do the Iwasawa invariants vary as p varies over all odd primes p at which E has good reduction?

(ii) For a fixed prime p and fixed number field F, how do the Iwasawa invariants vary as E varies over all elliptic curves (with good reduction at p)?

(iii) Fix an elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ with good reduction at p. How do the Iwasawa invariants of $\mathsf{E}_{/F}$ vary when F varies over a family of number fields?

4.1. **Iwasawa invariants in $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$.** In [34], we started exploring questions at the intersection of arithmetic statistics and Iwasawa theory using the *Euler characteristic*. The Euler characteristic is defined as an alternating product of Galois cohomology groups. By the work of P. Schneider (see [50, 51]) and B. Perrin-Riou (see [44]), this invariant is known to be given by the *p-adic BSD formula* for primes of good ordinary reduction. Thus, it captures information about the size of the *Tate–Shafarevich group*, the *Tamagawa number*, the *anomalous primes*, and the (global) *torsion points* of the elliptic curve; thereby providing information about the Iwasawa invariants.

4.1.1. As a first step, Theorem 3.1 was extended to *higher rank* elliptic curves and to *supersingular primes*.

When E has *supersingular reduction* at p, the p-primary Selmer group, $\mathrm{Sel}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$, is *not* Λ-cotorsion. This makes the analysis of the algebraic structure of the Selmer group particularly difficult. Instead, one considers the *plus and minus* Selmer groups, denoted by $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$, which were introduced by S. Kobayashi in [21] and are known to be Λ-cotorsion. The Iwasawa invariants $\mu^{\pm}$ and $\lambda^{\pm}$ associated with the $\pm$-Selmer group are defined analogously. Even in the supersingular case, there is sufficient computational evidence suggesting that *often* the associated μ-invariants vanish. Under standard hypotheses on the Shafarevich–Tate group, it is easy to show that the λ-invariant associated to a (Λ-cotorsion) Selmer group is *always* at least as large as the Mordell–Weil rank of E. For rank 0 elliptic curves over $\mathbb{Q}$, the precise $\pm$-Euler characteristic formula (associated to $\pm$-Selmer groups) has been obtained by Lei and Sujatha in [39]. This recent result was used crucially in proving the following theorem.

Theorem 4.2. *Let $\mathsf{E}_{/\mathbb{Q}}$ be a fixed rank 0 elliptic curve. Assume finiteness of the Shafarevich–Tate group over $\mathbb{Q}$. Then, for* all but finitely many *primes primes of good supersingular reduction* $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$ *is trivial. In particular,* $\mathrm{Sel}_0(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$ *is trivial for all but finitely many supersingular primes.*

The final assertion holds because for a prime of supersingular reduction, $\mathrm{Sel}_0(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$ is a subgroup of $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$. Combining Theorems 3.1 and 4.2, the next result is immediate as there are only a finite number of bad primes for E.

Corollary 4.3. *Let $\mathsf{E}_{/\mathbb{Q}}$ be a fixed rank 0 elliptic curve. Assume finiteness of the Shafarevich–Tate group over $\mathbb{Q}$. Then, Conjecture A holds for density 1 primes.*

In the *higher rank* setting, answering the question is more difficult. This is because of the presence of the (normalized) p-adic regulator term in the Euler characteristic formula. In any case, the following theorem can be proven.

Theorem 4.4. *Let $\mathsf{E}_{/\mathbb{Q}}$ be an elliptic curve such that its Mordell–Weil rank, $r_{\mathsf{E}} \geq 1$. Then there exists an explicitly determined set of good ordinary primes such that $\mu(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$ and $\lambda(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = r_{\mathsf{E}}$.*

Remark 4.5.
(i) Numerical data suggests that this explicitly determined set from the above theorem is a *density 1 subset of the set of good ordinary primes*. The obstruction in attaining an unconditional result is due to our limited knowledge on how often the normalized p-adic regulator is a unit.
(ii) An analogue of Theorem 4.4 in the supersingular setting can be proven analogously, provided one assumes the *conjectural* Euler characteristic formula in this setting.

In the direction of Question 4.1(ii), the following result from [34] allows distinguishing between when the λ-invariant is *exactly equal* to the Mordell–Weil rank and when it is *strictly greater* than the rank.

Theorem 4.6. *Let $p \geq 5$ be a fixed prime number. Let $\mathcal{E}(X)$ be the set of isomorphism classes of all elliptic curves over $\mathbb{Q}$ with height $\leq X$. Let $\mathcal{J}(X)$ be the subset of $\mathcal{E}(X)$ containing rank 0 elliptic curves E with good reduction at p, and $\mathcal{Z}(X)$ be a subset for which either of the following hold:*

(i) if E has good ordinary reduction at p, then $\mathrm{Sel}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$ or
(ii) if E has good supersingular reduction at p, then $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}^{\mathrm{cyc}}) = 0$.

Then,

$$\limsup_{X\to\infty} \frac{\#\mathcal{Z}(X)}{\#\mathcal{E}(X)} \geq \limsup_{X\to\infty} \frac{\#\mathcal{J}(X)}{\#\mathcal{E}(X)} - \epsilon(p).$$

Here, $\epsilon(p)$ is an explicitly determined positive *constant which approaches 0 (quickly) as $p \to \infty$.*

On average, the proportion of elliptic curves over $\mathbb{Z}_p$ with good reduction at p (ordered by height) is $(1 - \frac{1}{p})$, see [5]. By Goldfeld's Conjecture, it is expected that 1/2 the elliptic curves have rank 1. Therefore, one expects that

$$\limsup_{X\to\infty} \frac{\#\mathcal{J}(X)}{\#\mathcal{E}(X)} = \frac{1}{2}\left(1 - \frac{1}{p}\right).$$

Theorem 4.6 indicates that for a *positive proportion* of elliptic curves $\mathrm{Sel}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$; the proportion approaches 1/2 as $p \to \infty$. In [35], it was possible to refine the results and prove that given any integer n, there is an *explicit lower bound* for the density of the set of elliptic curves with good ordinary reduction at p for which $\lambda + \mu \geq n$. This lower bound depends on p (and n), is *strictly positive*, and becomes smaller as p or n become larger. More precisely,

Theorem 4.7. *Let $n > 0$ be an integer and p be an odd prime number. Assume that the Shafarevich–Tate group is finite for all elliptic curves $\mathsf{E}_{/\mathbb{Q}}$. The set of elliptic curves $\mathsf{E}_{/\mathbb{Q}}$ with good* ordinary *reduction at p* and *the additional property that $\mu + \lambda \geq n$, has* positive density *which can be explicitly determined.*

4.2. **Iwasawa invariants in anti-cyclotomic $\mathbb{Z}_p$-extensions.** In [13], the goal was to study problems raised in Question 4.1 for rank 0 elliptic curves with good *ordinary* reduction at p over the *anti-cyclotomic* $\mathbb{Z}_p$-extensions of an *imaginary quadratic field* in both the *definite* and the *indefinite* setting.

4.2.1. **Definite Case: Heegner hypothesis is *not* satisfied.** This setting was studied by R. Pollack and T. Weston in [47], and in their joint work with C.-H. Kim [18]. Here, the number of bad inert primes is odd, preventing the existence of Heegner points. Consequently, there should be few rational points. Their work confirms this and shows that under various hypotheses, the anti-cyclotomic Selmer group is cotorsion. Hence, the story is somewhat similar to the cyclotomic one. In particular, it is possible to prove an Euler characteristic formula for $\mathrm{Sel}(\mathsf{E}/K_{\mathrm{ac}})$.

In the direction of Question 4.1(i), it is possible to prove that for *non-CM elliptic curves*, the exact order of growth for the number of primes at which $\mu = 0$ is closely related to the Lang–Trotter Conjecture.

In response to Question 4.1(ii) it can be shown that for rank 0 elliptic curves the answer is primarily dependent on the variation of Shafarevich–Tate groups, which can be studied via the heuristics of C. Delaunay.

Question 4.1(iii) however is more subtle. This question is largely dependent on the divisibility by p of the order of the Shafarevich–Tate group upon base-change to $\mathbb{Q}(\sqrt{-d})$ (as $d > 0$ varies). Even though it appears difficult to provide (unconditional) theoretical results, there is computational data which suggests that "often" large values of p *do not* divide the order of the Shafarevich–Tate group.

Refinements in the case of supersingular reduction. In [36], we revisited Question 4.1(iii) with F. Sprung. The goal of this paper was to understand the average behaviour of Iwasawa invariants at primes of *supersingular reduction*. The key difference of this work from [13] is that it *does not use* the Euler characteristic formula. Instead, it focuses on measuring how mild the assumptions in [18] and [47] are from a statistical point of view. A bit more precisely, it asserts that the proportion of such imaginary quadratic fields is halved for each prime of bad reduction that is *split* that would violate the key hypothesis of [18] *were it inert*. The main result is the following.

Theorem 4.8. *Fix a pair* $(\mathsf{E}_{/\mathbb{Q}}, p)$ *so that*

(*i*) $\mathsf{E}_{/\mathbb{Q}}$ *is an elliptic curve with square-free conductor* $N_{\mathsf{E}} = \prod_{i=1}^{r} q_i$, *and*
(*ii*) $p > 3$ *is a prime at which* E *has good* supersingular *reduction,* $\overline{\rho}_{\mathsf{E},p}$ *is surjective, and* $k < r$.

Then the proportion of imaginary quadratic fields such that $\gcd\left(\left|\operatorname{disc}\mathbb{Q}(\sqrt{-d})\right|, pN_{\mathsf{E}}\right) = 1$, *the prime* p *splits in* $\mathbb{Q}(\sqrt{-d})$, *and* $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}(\sqrt{-d})_{\mathrm{ac}})$ *with associated* μ*-invariant equal to zero is at least is* Λ*-cotorsion is* at least

$$\frac{pN_{\mathsf{E}}}{2^{k+2}(p+1)\prod_{q_i \mid N_{\mathsf{E}}}(q_i+1)} \cdot \left(1 - c_p^*\right).$$

Here, the constant c_p^* *is related to the Cohen–Lenstra heuristics.*

Remark 4.9. An analogous result can be proven in the good *ordinary* setting as well. The only difference is that p can be either split or inert in $\mathbb{Q}(\sqrt{-d})$. So, the proportion is doubled.

4.2.2. **Indefinite Case: Heegner hypothesis holds.** When the Heegner hypothesis holds, the p-primary Selmer group $\mathrm{Sel}(\mathsf{E}/K_{\mathrm{ac}})$ is *not* Λ-cotorsion. The theory in this setting is vastly different. Many of the arguments used in proving the earlier results fail, and their analogues are often false. Importantly, in this setting, there is no known formula for the Euler characteristic of $\mathrm{Sel}(\mathsf{E}/K_{\mathrm{ac}})$. This issue can be circumvented by relating this Selmer group to an *auxiliary* Selmer group which is Λ-cotorsion and then using recent progress towards the anti-cyclotomic Iwasawa Main Conjectures made by A. Burungale–F. Castella–C.-H. Kim [3] to obtain an Euler characteristic formula for the auxiliary Selmer group. It appears that answering Questions 4.1(i)-(iii) systematically in the indefinite setting is deeply intertwined with the theory of the *BDP* p*-adic* L*-function* and is currently out of reach. However, it was possible to provide some partial answers and supplement the results with computational data.

4.3. **Iwasawa invariants in non-commutative p-adic Lie extensions.** In [29], which is joint work with A. Lei and A. Ray, we have extended the study of average Iwasawa invariants to the non-commutative setting. What makes the task of determining Iwasawa invariants in this situation more challenging is that in a non-commutative p-adic Lie extension F_∞/F, it is possible for primes other than p to ramify. Given a triple $(\mathsf{E}, p, F_\infty)$, the main task in this article was study the variation of the algebraic structure of the Selmer group $\mathrm{Sel}(\mathsf{E}/F_\infty)$ in three different contexts.

(a) Fix the pair (E, p) and let F_∞ vary over a family of admissible extensions.
(b) Fix the pair (p, F_∞) and let E vary over a subset of elliptic curves $\mathsf{E}_{/\mathbb{Q}}$ of rank 0.
(c) Fix an elliptic curve E and associate to each prime p, an extension F_∞ in a natural way. Then, vary p over the primes at which E has good *ordinary* reduction.

Using crucially the Euler characteristic formula, we studied these three questions in three distinct settings.

(1) First, consider the $\mathbb{Z}_p^2$-extension of imaginary quadratic fields. This is a 2-dimensional *abelian* extension and a *metabelian* extension over $\mathbb{Q}$. This case parallels the cyclotomic theory and in fact, the Euler characteristics for the $\mathbb{Z}_p^2$-extension and the cyclotomic $\mathbb{Z}_p$-extension coincide.

(2) The next step is to work with the simplest non-commutative 2-dimensional p-adic Lie extension, which is the false Tate curve extension. Given primes p and ℓ, write

$$F_\infty := \mathbb{Q}(\mu_{p^\infty}, \ell^{\frac{1}{p^n}} : n = 1, 2, \cdots)$$

and set $G = \operatorname{Gal}\left(F_\infty/F\right) \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_p$. The three questions of interest in this case can be explained as follows:

(a) Fix an elliptic curve E of conductor N_{E} and a prime p of good *ordinary* reduction of E. Consider the family of false Tate curve extensions obtained by varying $\ell \nmid N_{\mathsf{E}}p$. The first question involves studying for what proportion of primes ℓ is the Selmer group trivial over F_∞.

(b) Fix the primes p and ℓ, and let E vary over all elliptic curves defined over $\mathbb{Q}$ ordered by height. The next question is answered by calculating an upper bound for the proportion of elliptic curves for which the Selmer group is *not* trivial.

(c) Finally, fix a rank 0 elliptic curve $\mathsf{E}_{/\mathbb{Q}}$, a good prime ℓ, and let p vary over the primes at which E has good *ordinary* reduction. It is possible to show that for *at least half* of the primes p, the G-Euler characteristic coincides with the Γ-Euler characteristic. When E has good *supersingular* reduction at ℓ, this happens for *exactly two-third* of the primes p. For such primes p, the Selmer group over the false Tate curve extension is trivial if and only if that over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}(\mu_p)$ is trivial.

(3) Finally, it is worth understanding the case of the trivializing extension $F_\infty/\mathbb{Q}$, generated by the p-primary torsion points of a non-CM elliptic curve (denoted by A, E_0 and E' in the three questions). Since $\mathcal{G} := \operatorname{Gal}\left(F_\infty/\mathbb{Q}\right)$ is *not* a pro-p extension, it is not possible to exploit the theory of Akashi series and translate the results on the $\mathcal{G}$-Euler characteristic formula to conclusions regarding the pseudonullity of the Selmer group over the infinite extension. The main results in this direction are the following:

(a) Fix a rank 0 elliptic curve E of conductor N_{E} and a prime p of good *ordinary* reduction of E. There is a family of extensions obtained by varying a non-CM elliptic curve $A_{/\mathbb{Q}}$. For density 0 (but infinitely many) such elliptic curves A, the $\mathcal{G}$-Euler characteristic is trivial.

(b) Fix p and a non-CM elliptic curve $\mathsf{E}_{0/\mathbb{Q}}$. This fixes the p-adic Lie extension $\mathbb{Q}\left(\mathsf{E}_{0,p^\infty}\right)/\mathbb{Q}$. As E varies over all elliptic curves defined over $\mathbb{Q}$ and ordered by height, it is possible to calculate an upper bound for the proportion of elliptic curves for which the $\mathcal{G}$-Euler characteristic is *not* trivial.

(c) For a pair of elliptic curves $(\mathsf{E}, \mathsf{E}')$ such that E' does not have CM, consider the Selmer group of E over the p-adic Lie extension $\mathbb{Q}\left(\mathsf{E}'[p^\infty]\right)/\mathbb{Q}$ as p varies. For all but finitely many primes, the $\mathcal{G}$-Euler characteristic is equal to the Γ-Euler characteristic. Moreover, [34, Conjecture 3.17] predicts that this latter quantity is trivial *most of the time*.

4.4. **Diophantine stability.** Questions pertaining to rank growth are of much interest to arithmetic geometers. In [2], the aim was to study questions pertaining to Diophantine stability using tools and techniques from Iwasawa theory. To answer questions on rank jump of elliptic curves upon base change, the natural thing to do was to study the growth of a more tractable arithmetic object, i.e., the p-primary Selmer groups upon base change. In particular, the following two questions were investigated using a Kida-type formula for λ-invariants (proven in [10]).

Question 4.10. (i) Given an elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ with trivial p-primary Selmer group, for what proportion of degree-p cyclic extensions does the p-primary Selmer group remain trivial upon base-change.

(ii) Given $p \neq 2, 3$, for how many elliptic curves over $\mathbb{Q}$ does there exist *at least one* $\mathbb{Z}/p\mathbb{Z}$-extension where the p-primary Selmer group remains trivial upon base-change.

The answer to the first question is the following.

Theorem 4.11. *Given an elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ and a prime $p \geq 7$ with $\mu(\mathsf{E}/\mathbb{Q}_{\text{cyc}}) = \lambda(\mathsf{E}/\mathbb{Q}_{\text{cyc}}) = 0$, there are infinitely many $\mathbb{Z}/p\mathbb{Z}$-extensions of $\mathbb{Q}$ where the λ-invariant does not increase; in particular, the rank does not jump. Moreover, there are infinitely many $\mathbb{Z}/p\mathbb{Z}$-extensions of $\mathbb{Q}$ where the Mordell–Weil group does not grow.*

The assertion on rank growth follows from the fact that the Mordell–Weil rank is *at most* as large as the λ-invariant. The final assertion is an immediate consequence of a recent result of [7] on torsion growth. The proof shows that the λ-invariant does not jump in *many* $\mathbb{Z}/p\mathbb{Z}$-extensions. Unfortunately, the method falls short of proving a positive proportion as predicted by a conjecture of C. David–J. Fearnley–H. Kisilevsky.

A natural follow-up question is when does the p-primary Selmer group grow upon base-change. After establishing a criterion for either the rank to jump, or the order of the Shafarevich–Tate group to increase upon base-change, the next result is proven by exploiting the relationship between λ-invariants and the Euler characteristic formula.

Theorem 4.12. *Let $p \geq 5$ be a fixed prime and $\mathsf{E}_{/\mathbb{Q}}$ be an elliptic curve with good ordinary reduction at p. Suppose that the image of the residual representation is surjective,* $\mathrm{Sel}(\mathsf{E}/\mathbb{Q})$ *is trivial, and $\mu(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = \lambda(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$. Then, there is a set of primes of the form $q \equiv 1 \pmod{p}$ with density at least $\frac{p}{(p-1)^2(p+1)}$ such that the p-primary Selmer group* becomes non-trivial *in the unique $\mathbb{Z}/p\mathbb{Z}$-extension contained in $\mathbb{Q}(\mu_q)$.*

The following theorem answers Question 4.10(ii).

Theorem 4.13. *For a* positive proportion *of rank 0 elliptic curves defined over $\mathbb{Q}$, there exists* at least one $\mathbb{Z}/p\mathbb{Z}$-*extension over $\mathbb{Q}$ disjoint from $\mathbb{Q}_{\mathrm{cyc}}$, such that the p-primary Selmer group upon base-change is trivial.*

4.5. **Hilbert's 10th Problem.** In [30], which is joint work with A. Lei and F. Sprung, we studied the analogue of Hilbert's 10th Problem for rings of integers of number fields, which asks the following question:

Is $\mathbb{Z}$ a Diophantine subset of the ring of integers of a number field L?

The strategy to prove such results is inspired by the work of N. Garcia-Fritz–H. Pasten, see [6]. However, there are some important differences which will be explained after mentioning the key ingredients that go into the proof.

Main idea of the proof: The proof has the following three main steps:
Step 1: The following assertion is a straightforward corollary of a result by A. Shlapentokh: if there exists a rank 0 elliptic curve over $\mathbb{Q}$ such that in the quadratic extension $K/\mathbb{Q}$ the rank jumps, and in the extension $F/\mathbb{Q}$ the rank remains 0, then Hilbert's 10th problem has a *negative solution* for the composite number field, $L = K.F$.

If $F/\mathbb{Q}$ is integrally Diophantine then by the Transitivity Property, $L/\mathbb{Q}$ is also integrally Diophantine.

Step 2: Find a rank 0 elliptic curve $E/\mathbb{Q}$ satisfying certain mild conditions and two families of number fields:

- a family (of cubic number fields) such that rank of $E/\mathbb{Q}$ does not jump.
- another family of (quadratic) extensions such that rank jumps.

Step 3: Determine 'how big' these families are. To count the number field extensions where the rank is stabilized, the authors of [6] crucially used Iwasawa theory. As is shown in [30], it is possible to provide a direct argument. To count the number of quadratic extensions with rank jump requires the work of D. Kriz–C. Li [22] (or the work of A. Smith [53] when working with the congruent number curves).

Improving the results of Garcia-Fritz–Pasten. Here are the main differences between this recent work and [6]:

First, using a result from a previous work [29, Section 8 and Appendix] it is possible to refine the results of [6] and improve the densities of both $\mathcal{P}$ and $\mathcal{Q}$.

Theorem 4.14. *There are explicit Chebotarev sets of primes $\mathcal{P}$ and $\mathcal{Q}$, of density $\frac{9}{16}$ and $\frac{7}{48}$, such that for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, the analogue of Hilbert's 10th Problem is unsolvable for the ring of integers of $L = \mathbb{Q}(\sqrt[3]{p}, \sqrt{-q})$.*

Second, it was possible to provide a direct proof of the main theorem of [6] by proving the vanishing of certain 3-Selmer groups (rather than the finiteness of 3^∞-Selmer group). This allows significant weakening of the hypotheses and the possibility to find many auxiliary elliptic curves (not necessarily of positive minimal discriminant).

Theorem 4.15. *Let*

$$\mathfrak{D} = \{7, 39, 95, 127, 167, 255, 263, 271, 303, 359, 391, 447, 479, 527, 535, 615, 623, 655, 679, 695\}.$$

For all $D \in \mathfrak{D}$, there are explicit Chebotarev sets of primes $\mathcal{P}$ (independent of D) and $\mathcal{Q}_D$, of density $\frac{9}{16}$ and $\frac{1}{12}$ such that for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}_D$, the analogue of Hilbert's 10th Problem is unsolvable for the ring of integers of $L = \mathbb{Q}(\sqrt[3]{p}, \sqrt{Dq})$.

By working with a pair of auxiliary elliptic curves, it is possible to improve significantly the density of $\mathcal{P}$, at the expense of a smaller set of $\mathfrak{D}$ and a lower density for the sets $\mathcal{Q}_D$.

Theorem 4.16. *Let $D \in \{7, 615\}$. There are explicit Chebotarev sets of primes $\mathcal{P}$ (independent of D) and $\mathcal{Q}_D$, of density $\frac{103}{128}$ and $\frac{1}{36}$, such that for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}_D$, the analogue of Hilbert's 10th problem is unsolvable for the ring of integers of $L = \mathbb{Q}(\sqrt[3]{p}, \sqrt{Dq})$.*

Finally, the more direct approach permits working with elliptic curves with good supersingular reduction at 3. This provide the opportunity to use the congruent number curve.

Theorem 4.17. *There is an explicit Chebotarev set of primes $\mathcal{P}$ with density $\frac{11}{16}$ such that Hilbert's 10th Problem is unsolvable for the ring of integers of $L = \mathbb{Q}(\sqrt[3]{p}, \sqrt{q})$ whenever q is a congruent number.*

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