ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 51 (2021), No. 3, 933–949

PERFECT POWERS THAT ARE SUMS OF SQUARES OF AN ARITHMETIC PROGRESSION

DEBANJANA KUNDU AND VANDITA PATEL

We determine all nontrivial integer solutions to the equation $(x+r)^2+(x+2r)^2+\cdots+(x+dr)^2=y^n$ for $2\le d\le 10$ and $1\le r\le 10^4$ with $\gcd(x,y)=1$. We make use of a factorization argument and the primitive divisors theorem due to Bilu, Hanrot and Voutier.

1. Introduction

Finding perfect powers that are sums of powers of terms in an arithmetic progression has received much interest; recent contributions can be found in [2; 3; 4; 5; 6; 7; 8; 9; 13; 15; 16; 17; 20; 21; 22]. We consider the equation

$$(x+r)^2+(x+2r)^2+\cdots+(x+dr)^2=y^n \quad x,y,n\in\mathbb{Z},\ \gcd(x,y)=1,\ n\ge 2, \tag{1}$$

where d is a fixed positive integer and r is a positive integer. We note in passing that the condition $\gcd(x,y)=1$ in (1) is equivalent to x,y,r being pairwise coprime in (1). We say that a solution is trivial if $xy=0$ and nontrivial otherwise. We prove the following theorem:

Theorem 1.1. *For $d\in\{4,5,7,8,9,10\}$ and any positive integer r, (1) has no nontrivial integer solutions.*

Let $d\in\{2,3,6\}$ and $1\le r\le 10^4$. All nontrivial solutions to (1) for $d=2$ with prime exponent $n\ge 3$ are given in Section 8A, and with exponent $n=4$ are given in Section 8B. For $d=3$, all nontrivial solutions are given in [14] and for $d=6$, all nontrivial solutions are recorded in Section 8C.

When $d=2$ and $n=2$, we have no nontrivial solutions unless every prime divisor of r is congruent to $\pm1 \pmod 8$. Suppose we are in the latter case, let $r=q_1^{t_1}\cdots q_s^{t_s}$ where the q_i are distinct primes. For each i we may write $q_i=\mathrm{Norm}(\mathfrak{q}_i)$ with $\mathfrak{q}_i\in\mathbb{Z}[\sqrt{2}]$. Then the solutions are given by

$$2x+3r+y\sqrt{2}=\pm\mathfrak{r}^2\cdot(1+\sqrt{2})^{2k+1},\quad k\in\mathbb{Z}$$

where $\mathfrak{r}=\mathfrak{r}_1^{t_1}\cdots\mathfrak{r}_s^{t_s}$ and each $\mathfrak{r}_i$ is either $\mathfrak{q}_i$ or its conjugate $\overline{\mathfrak{q}_i}$.

Cohn [12] solves (1) for $d=2$, $r=1$, $n\ge 3$ and as an application, finds all perfect powers in the Pell sequence. In light of the considerable recent interest in finding perfect powers that are sums of three terms in arithmetic progression, Patel, with Koustianas [14], list all nontrivial solutions to (1) for the case $d=3$ and prime exponent n with $1\le r\le 10^4$. As a natural extension to [14], we consider (1) for $2\le d\le 10$. Theorem 1.1 lists our findings.

2020 AMS *Mathematics subject classification:* 11D61.
Keywords and phrases: exponential equation, Lehmer sequence, primitive divisor, Thue equation.
Received by the editors on July 14, 2020.

DOI: 10.1216/rmj.2021.51.933

In the proof of Theorem 1.1, we are able to solve the special case that Cohn studied ($d = 2$, $r = 1$, $n \geq 3$) uniformly with other values of r when $d = 2$. We use a different approach to that taken in Cohn's original paper [12]. We have also consolidated the results of [14] into Theorem 1.1, which studies the case $d = 3$.

Remark 1.2. (1) Nontrivial solutions to (1) with exponent n that is composite can be recovered from Sections 8A, 8B and 8C by checking whether y is a perfect power.

(2) For $d = \{2, 3, 6\}$, we have the restriction $1 \leq r \leq 10^4$ due to a computational limitation. This arises due to the presence of extremely large coefficients of certain polynomials and certain Thue equations. The methodology is applicable to general values of r.

(3) Values of d where all prime divisors of d are congruent to $\pm 1 \pmod{12}$ are not amenable to the techniques developed in this paper, hence we have the restriction $2 \leq d \leq 10$.

General theorems on equations of the form $x^2 + C = 2y^p$ can be found in [1; 18; 19]. Our main tool is the characterization of primitive divisors in Lehmer sequences due to Bilu, Hanrot and Voutier [10], along with solving some Thue equations for small values of n. Tables of solutions are located in Section 8.

2. Some precursory lemmata

In this section, we adapt Lemmas 2.2 and 2.3 from [15] and Lemma 2.3 from [16]. We continue to work under the assumption $\gcd(x, y) = 1$, which we recall is equivalent to x, y, r being pairwise coprime. Let d and r be positive integers and $n \geq 2$.

We rewrite (1) as

$$dx^2 + d(d+1)xr + \frac{d(d+1)(2d+1)}{6}r^2 = y^n. \tag{2}$$

Factoring and completing the square gives us

$$d\left(\left(x + \frac{d+1}{2}r\right)^2 + \frac{(d-1)(d+1)}{12}r^2\right) = y^n. \tag{3}$$

Observe in particular that $y \neq 0$.

Lemma 2.1. *Let $j = \operatorname{ord}_2(d)$. If $j \geq 2$, then in (3) we have $n \mid (j-1)$.*

Proof. Let $D = d/2^2$. We substitute into (3) to get,

$$D\left((2x + (d+1)r)^2 + \frac{(d-1)(d+1)}{3}r^2\right) = y^n. \tag{4}$$

Since $j \geq 2$, (2) shows that $2 \mid y$ and therefore, $2 \nmid r$ since $\gcd(x, y) = 1$. Observe that

$$(2x + (d+1)r)^2 \equiv 1 \pmod 4, \qquad \frac{(d-1)(d+1)}{3}r^2 \equiv 1 \pmod 4.$$

Comparing valuations on both sides of (4) we see that

$$n \operatorname{ord}_2(y) = \operatorname{ord}_2(D) + 1 = j - 1.$$

This completes the proof. $\square$

Lemma 2.2. *Let* $j = \mathrm{ord}_3(d)$. *If* $j \geq 2$, *then in* (3), *we have* $n \mid (j-1)$.

Proof. Let $D = d/3$. We substitute into (3) to get,

$$D\left(3\left(x+\frac{(d+1)}{2}r\right)^2+\frac{(d-1)(d+1)}{4}r^2\right)=y^n.$$

Since $j \geq 2$, (2) asserts that $3 \mid y$ and therefore $3 \nmid r$ since we assume throughout that $\gcd(y, r) = 1$. Observe that the expression in brackets is never divisible by 3. Hence $\mathrm{ord}_3(D) = \mathrm{ord}_3(y^n) = n\,\mathrm{ord}_3(y)$, thus proving the lemma. □

Lemma 2.3. *Let* r *be a nonzero positive integer. Let* q *be a prime such that* $q \equiv \pm 5 \pmod{12}$. *Let* d *be a positive integer such that* $\mathrm{ord}_q(d) \not\equiv 0 \pmod{n}$. *Then* (1) *has no solutions.*

Proof. Our assumption on q forces $q \neq 2, 3$, and (2) affirms that $q \mid y$. Since $\gcd(y, r) = 1$, $q \nmid r$. As $d \equiv 0 \pmod{q}$ and $\mathrm{ord}_q(d) \not\equiv 0 \pmod{n}$, (3) tells us that

$$\left(x+\frac{r}{2}\right)^2 \equiv \frac{1}{12} \pmod{q}.$$

This implies $q \equiv \pm 1 \pmod{12}$ which gives a contradiction. □

Applying Lemmata 2.1, 2.2 and 2.3 allows us to prove that for $d \in \{4, 5, 7, 9, 10\}$, with $n \geq 2$ and r a positive integer, or for $d = 8$ with $n \geq 3$ and r a positive integer, (1) has no nontrivial solutions. In order to complete the proof of Theorem 1.1, it remains to deal with $d = 2, 3, 6$ for $n \geq 2$, and also with $d = 8$ for $n = 2$. The case $d = 3$, $1 \leq r \leq 10^4$ for $n \geq 2$ has been resolved in [14] and a table of solutions can be found in that paper.

3. Case $n = 2$

In this section, we deal with the case $n = 2$ when $d = 2, 6, 8$.

Lemma 3.1. *Let* $d = 6$ *or* 8 *and* $n = 2$. *Then* (1) *has no integer solutions.*

Proof. Let $d = 6$, $n = 2$. We rewrite (1) as

$$3(2x+7r)^2+35r^2=2y^2.$$

As 6 is a nonsquare modulo 7, we see that $7 \mid (2x+7r)$ and $7 \mid y$ which quickly contradicts $\gcd(x, y) = 1$.

When $d = 8$, $n = 2$, we rewrite (1) as

$$2((2x+9r)^2+21r^2)=y^2.$$

Writing $y = 2Y$ we obtain

$$(2x+9r)^2+21r^2=2Y^2$$

and considering the equation modulo 3, we see that 2 must be a square modulo 3 and arrive at a contradiction. □

We finally look at the case $d = 2$, $n = 2$. Here we prove the claim made about this case in the statement of Theorem 1.1.

Proof of Theorem 1.1 for $d = 2$, $n = 2$. In this case we can rewrite (1) as

$$2x^2 + 6xr + 5r^2 = y^2.$$

We immediately notice that r is odd (otherwise we contradict $\gcd(y, r) = 1$). We may rewrite this as

$$(2x + 3r)^2 - 2y^2 = -r^2. \tag{5}$$

It follows that 2 is a quadratic residue modulo any prime divisor of r and so they are all of the form ± 1 mod 8, and so split in $\mathbb{Z}[\sqrt{2}]$. Write $r = q_1^{t_1} \cdots q_s^{t_s}$ as in the theorem where the q_i are distinct primes. For each i let $\mathfrak{q}_i \in \mathbb{Z}[\sqrt{2}]$ satisfy $\mathrm{Norm}(\mathfrak{q}_i) = q$; these $\mathfrak{q}_i$ are primes of $\mathbb{Z}[\sqrt{2}]$. Thus the prime divisors of $2x + 3r + y\sqrt{2}$, $2x + 3r - y\sqrt{2}$ are among $\mathfrak{q}_i$, $\overline{\mathfrak{q}_i}$. Since $\gcd(x, y) = 1$, $2x + 3r + y\sqrt{2}$, $2x + 3r - y\sqrt{2}$ are coprime in $\mathbb{Z}[\sqrt{2}]$. Thus $2x + 3r + y\sqrt{2}$ is divisible by either $\mathfrak{q}_i$ or $\overline{\mathfrak{q}_i}$ but not both, and moreover, the valuation at this prime is $2t_i$. Thus $2x + 3r + y\sqrt{2} = \epsilon \cdot \mathfrak{r}^2$ where ϵ is a unit, and $\mathfrak{r}$ is as in the statement of the theorem. Taking norms and comparing to (5) we see that $\mathrm{Norm}(\epsilon) = -1$, so $\epsilon = \pm(1 + \sqrt{2})^{2k+1}$ for some integer k. This completes the proof in this case. □

Remark 3.2. As we see infinitely many solutions arising in the case $d = 2$, $n = 2$, it remains to solve (1) with $n = 4$. This will be done in the next section.

4. Case $n = 4$

In this section, we find all integer solutions to the equation

$$(x + r)^2 + (x + 2r)^2 = y^4.$$

We note that since $\gcd(x, r) = 1$ we must have $\gcd(x + r, x + 2r) = 1$. We denote $\sqrt{-1} = i$. Applying a descent argument over the Gaussian integers, we obtain

$$(x + 2r) + i(x + r) = \epsilon\alpha^4 \tag{6}$$

where $\epsilon \in \{\pm 1, \pm i\}$ is a unit and $\alpha \in \mathbb{Z}[i]$. We let $\alpha = u + iv$, where $u, v \in \mathbb{Z}$.

Case 1:The unit $\epsilon = \pm 1$. We equate real and imaginary parts of (6) to obtain the equations

$$x + 2r = \pm(u^4 - 6u^2v^2 + v^4),$$
$$x + r = \pm 4(u^3v - uv^3).$$

Subtracting one from the other, we get

$$\pm r = u^4 - 4u^3v - 6u^2v^2 + 4uv^3 + v^4.$$

Case 2:The unit $\epsilon = \pm i$. Equating real and imaginary parts of (6), we obtain

$$x + 2r = \pm 4(-u^3v + uv^3),$$
$$x + r = \pm(u^4 - 6u^2v^2 + v^4).$$

Subtracting one from the other, we get

$$\pm r = u^4 + 4u^3v - 6u^2v^2 - 4uv^3 + v^4.$$

In both cases, when r has a fixed value, we obtain homogeneous equations of degree 4. Using `Magma's` Thue solver, we determine all integer solutions (u, v), whereby we recover the nontrivial integer solutions $(x, |y|, n = 4)$ to (1) for $d = 2$. These are recorded in Section 8B.

5. Primitive prime divisors of Lucas and Lehmer sequences

A *Lehmer pair* is a pair of algebraic integers α, β, such that $(\alpha+\beta)^2$ and $\alpha\beta$ are nonzero coprime rational integers and α/β is not a root of unity. The *Lehmer sequence* associated to the Lehmer pair (α, β) is

$$\tilde{u}_n = \tilde{u}_n(\alpha, \beta) = \begin{cases} (\alpha^n - \beta^n)/(\alpha - \beta) & \text{if } n \text{ is odd,} \\ (\alpha^n - \beta^n)/(\alpha^2 - \beta^2) & \text{if } n \text{ is even.} \end{cases}$$

A prime p is called a *primitive divisor* of $\tilde{u}_n$ if it divides $\tilde{u}_n$ but does not divide $(\alpha^2 - \beta^2)^2 \cdot \tilde{u}_1 \cdots \tilde{u}_{n-1}$. We shall make use of the following celebrated theorem [10].

Theorem 5.1 (Bilu, Hanrot and Voutier). *Let α, β be a Lehmer pair. Then $\tilde{u}_n(\alpha, \beta)$ has a primitive divisor for all $n > 30$, and for all prime $n > 13$.*

6. An arithmetic progression with two terms

In this section, we find all nontrivial integer solutions to (1) for $d = 2$, $1 \le r \le 10^4$ and n an odd prime. We rewrite (1) as

$$2x^2 + 6xr + 5r^2 = y^n.$$

Multiplying by 2 and completing the square, we obtain

$$(2x + 3r)^2 + r^2 = 2y^n. \tag{7}$$

We apply the following general theorem.

Theorem 6.1 [1, Theorem 1]. *Let C be a positive integer satisfying $C \equiv 1 \pmod 4$ and write $C = cd^2$ where c is square-free. Suppose that (x, y) is a solution to the equation*

$$x^2 + C = 2y^p, \quad x, y \in \mathbb{Z}^+, \quad \gcd(x, y) = 1,$$

where $p \ge 5$ is a prime. Then either,

(i) $x = y = C = 1$, *or*

(ii) *p divides the class number of $\mathbb{Q}(\sqrt{-c})$, or*

(iii) *$p = 5$ and $(C, x, y) = (9, 79, 5), (125, 19, 3), (125, 183, 7), (2125, 21417, 47)$, or*

(iv) *$p \mid (q - (-c)/q)$, where q is some odd prime such that $q \mid d$ and $q \nmid c$.*

6A. Proof of Theorem 1.1 for $d = 2$. We rewrite (7) as

$$|2x + 3r|^2 + r^2 = 2y^n$$

and apply Theorem 6.1. Case (i) gives the solutions $x = -1$ or -2, $r = 1$, $y = 1$ and n arbitrary. We suppose we are not in this case. Let

$$B = \{3, 5\} \cup \left\{p \text{ odd prime} : p \mid \left(q - \left(\tfrac{-c}{q}\right)\right), \text{ for some odd prime } q \mid r\right\}.$$

Note that if $r = 1$ then $B = \{3, 5\}$. Theorem 6.1 asserts $n \in B$. Thus for every $2 \le r \le 10^4$ we have finitely many possible values of the prime exponent n. We will explain how to solve (7) for a fixed r and fixed exponent n. From (7) we obtain

$$2x + 3r + ir = (1+i)\alpha^n$$

for some $\alpha \in \mathbb{Z}[i]$. Subtracting this equation from its conjugate gives

$$(1+i)\alpha^n - (1-i)\bar{\alpha}^n = 2ri.$$

Dividing by $1+i$ we have

$$\alpha^n + i\bar{\alpha}^n = (1+i)r. \tag{8}$$

Let $\alpha = u + iv$ with $u, v \in \mathbb{Z}$. If $n \equiv 1 \pmod 4$ then $i = i^n$ and if $n \equiv -1 \pmod 4$ then $i = (-i)^n$. In the former case $\alpha + i\bar{\alpha} = (1+i)(u+v)$ is a factor of the left-hand side of (8), and in the latter case $\alpha - i\bar{\alpha} = (1-i)(u-v)$ is a factor. We deduce that $(u+v) \mid r$ or $(u-v) \mid r$ according to whether $n \equiv 1$ or $-1 \pmod 4$. Thus for each $1 \le r \le 10^4$ and for each $n \in B$ and for each $t \mid r$, we let $u \pm v = t$, and we need to simply solve for u, v. But (8) is now a polynomial equation in v after letting $\alpha = u + iv = (t \mp v) + iv$. We wrote a simple Magma script that solved these polynomial equations and deduced the corresponding solutions to (7). This gives the solutions (x, y, n) as in Section 8A.

7. An arithmetic progression with six terms

In this section, we find all nontrivial integer solutions to (1) for $d = 6$, $1 \le r \le 10^4$ and n an odd prime. We rewrite (1) as

$$X^2 + 3 \cdot 5 \cdot 7r^2 = 6y^n, \tag{9}$$

where we let $X = 6x + 21r$ for ease of notation. We note here that $2, 3 \nmid r$ else we contradict the assumption that $\gcd(x, y) = 1$. Let $K = \mathbb{Q}(\sqrt{-105})$ and its ring of integers, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-105}]$. This has class group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. We can factor (9) in $\mathcal{O}_K$ as follows

$$(X + r\sqrt{-105})(X - r\sqrt{-105}) = 6y^n.$$

Let us write $\mathfrak{p}_2$ and $\mathfrak{p}_3$ for the prime ideals above 2 and 3, respectively. Let $\mathfrak{a} = \mathfrak{p}_2\mathfrak{p}_3$. We write

$$(X + r\sqrt{-105})\mathcal{O}_K = \mathfrak{a}^{1-n} \cdot (\mathfrak{a}\mathfrak{z})^n = (6^{(1-n)/2})(\mathfrak{a}\mathfrak{z})^n,$$

where $\mathfrak{a}\mathfrak{z}$ is a principal ideal of $\mathcal{O}_K$. Indeed, $[\mathfrak{a}\mathfrak{z}]^n = 1$ in the class group. Therefore the class $[\mathfrak{a}\mathfrak{z}]$ has order dividing n or 1, as n is an odd prime. Since the class group has order 8, it means that the order of $[\mathfrak{a}\mathfrak{z}]$ must be 1. We therefore write $\mathfrak{a}\mathfrak{z} = (\gamma)\mathcal{O}_K$ where $\gamma = u + v\sqrt{-105} \in \mathcal{O}_K$ with $u, v \in \mathbb{Z}$. If required, we may swap γ with $-\gamma$ to obtain

$$X + r\sqrt{-105} = \frac{\gamma^n}{6^{(n-1)/2}}. \tag{10}$$

Subtracting the conjugate equation from the one above, we get

$$\frac{\gamma^n}{6^{(n-1)/2}} - \frac{\bar{\gamma}^n}{6^{(n-1)/2}} = 2r\sqrt{-105}, \tag{11}$$

or equivalently,

$$\frac{\gamma^n}{6^{n/2}} - \frac{\bar{\gamma}^n}{6^{n/2}} = r\sqrt{-70}.$$

Consider a quadratic extension, L/K, where $L = \mathbb{Q}(\sqrt{-105}, \sqrt{6}) = \mathbb{Q}(\sqrt{-70}, \sqrt{6})$. We write $\mathcal{O}_L$ for its ring of integers and set $\alpha = \gamma/\sqrt{6}$, $\beta = \bar{\gamma}/\sqrt{6}$. Thus (11) becomes

$$\alpha^n - \beta^n = r\sqrt{-70}. \tag{12}$$

Lemma 7.1. *Let α, β be as above. Then α and β are algebraic integers. Moreover, $(\alpha+\beta)^2$ and $\alpha\beta$ are nonzero, coprime, rational integers and α/β is not a unit.*

Proof. We observe that $\mathfrak{a} \cdot \mathcal{O}_L = \sqrt{6}\mathcal{O}_L$. By definition, $\mathfrak{a} \mid \gamma, \bar{\gamma}$ and hence α, β are algebraic integers. Let $\gamma = u + v\sqrt{-105}$ with $u, v \in \mathbb{Z}$. Then

$$(\alpha+\beta)^2 = \frac{2u^2}{3}.$$

Since $\mathfrak{p}_3 \mid \gamma, \sqrt{-105}$ we have $\mathfrak{p}_3 \mid u$ and so $3 \mid u$. Hence, $(\alpha+\beta)^2 \in \mathbb{Z}$. If $(\alpha+\beta)^2 = 0$ then $u = 0$. However, from (10) and the fact that n is odd, we obtain $X = 6x + 21r = 0$, hence $2x = -7r$. This contradicts the pairwise coprimality of x, y, r. Thus $(\alpha+\beta)^2$ is a nonzero rational integer. Moreover, $\alpha\beta = \gamma\bar{\gamma}/6$ is a nonzero rational integer since $3 \mid u$ and $\mathfrak{p}_2 \mid \gamma, \bar{\gamma}$.

We now check that $(\alpha+\beta)^2$ and $\alpha\beta$ are coprime. Suppose they are not coprime. Then there exists a prime $\mathfrak{q}$ of $\mathcal{O}_L$ which divides both. Then $\mathfrak{q}$ divides α, β and from (10),(9) and (12), we see that $\mathfrak{q}^n$ divides $(y^n)\mathcal{O}_L$ and $(r\sqrt{-70})\mathcal{O}_L$. Since $\operatorname{ord}_{\mathfrak{q}}(r\sqrt{-70}) \geq n$, with n an odd prime, we contradict our assumption of $\gcd(x, y) = 1$.

Finally, we show that $\alpha/\beta = \gamma/\bar{\gamma} \in \mathcal{O}_K$ is not a unit. If it were so, then since the units in K are ± 1 we obtain $\alpha = \pm\beta$. This implies that either $u = 0$ or $v = 0$. We have seen earlier that we cannot have $u = 0$. Substituting $v = 0$ into (10), we obtain $r = 0$ and again arrive at a contradiction. □

Lemma 7.1 tells us that (α, β) is indeed a Lehmer pair. We denote by $\tilde{u}_k$ the associated Lehmer sequence. We may rewrite (12) as

$$\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\left(\frac{\alpha - \beta}{\sqrt{-70}}\right) = r.$$

Hence, we have

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{r}{v} = r'. \tag{13}$$

Lemma 7.2. *Suppose $n > 13$. Then there is a prime $q \mid r$ such that $q \nmid 210$, and $n \mid B_q$ where*

$$B_q = \begin{cases} q-1 & \text{if } (-105/q) = 1, \\ q+1 & \text{if } (-105/q) = -1. \end{cases}$$

Proof. Let $n > 13$. By Theorem 5.1, $\tilde{u}_n = (\alpha^n - \beta^n)/(\alpha - \beta) = r'$ is divisible by a prime q not dividing $(\alpha^2 - \beta^2)^2 = -280u^2v^2/3$ nor the terms $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{n-1}$. We note that this is a prime q dividing r' but not $210v$. Let $\mathfrak{q}$ be a prime of K above q. As $(\alpha+\beta)^2$ and $\alpha\beta$ are coprime integers, and as α, β satisfy (13) we see that $\mathfrak{q} \nmid \gamma, \bar{\gamma}$. We make two claims:

(i) The multiplicative order of the reduction of $\gamma/\bar{\gamma}$ modulo $\mathfrak{q}$ is n.

(ii) The multiplicative order of the reduction of $\gamma/\bar{\gamma}$ modulo $\mathfrak{q}$ divides B_q.

It follows immediately that $n \mid B_q$ which is what we want to prove. We now need only prove (i), (ii). Let m be a positive integer. Note that $\alpha/\bar{\alpha} = \gamma/\bar{\gamma}$. Thus $q \mid \tilde{u}_m$ if and only if $(\gamma/\bar{\gamma})^m \equiv 1 \pmod{\mathfrak{q}}$. Thus (i) follows as q is a primitive divisor of $\tilde{u}_n$. Let us prove (ii). If -105 is a square modulo q, then $\mathbb{F}_{\mathfrak{q}} = \mathbb{F}_q$ and so the multiplicative order divides $q - 1 = B_q$. Otherwise, $\mathbb{F}_{\mathfrak{q}} = \mathbb{F}_{q^2}$. However, $\gamma/\bar{\gamma}$ has norm 1, and the elements of norm 1 in $\mathbb{F}_{q^2}^*$ form a subgroup of order $q + 1 = B_q$. In either case, the order divides B_q. □

7A. Proof of Theorem 1.1 for $d = 6$. Let

$$B = \{3, 5, 7, 11, 13\} \cup \left\{p \text{ odd prime} : p \mid \left(q - \left(\tfrac{-c}{q}\right)\right) \text{ for some odd prime } q \mid r\right\}.$$

Lemma 7.2 asserts that for (9), with $r \geq 1$, we have $n \in B$. We wrote a simple `Magma` [11] script which for each $1 \leq r \leq 10^4$ such that $2, 3 \nmid r$, and for each odd prime $v \mid r$, computed the set B. For each odd prime $n \in B$ we know from (11) that u is a root of

$$\frac{1}{2 \cdot r \cdot \sqrt{-105} \cdot 6^{(n-1)/2}} \cdot ((u + v\sqrt{-105})^n - (u - v\sqrt{-105})^n) - 1.$$

Computing these roots, we obtain the nontrivial solutions (x, y, n) as in Section 8C.

8. Tables of solutions

8A. Triples of nontrivial solutions (x, y, n) of (1) for $d = 2$ and prime $n \geq 3$ for $1 \leq r \leq 10^4$.

r	(x, y, n)
1	$(-1, 1, n)$, $(-2, 1, n)$
3	$(-41, 5, 5)$, $(38, 5, 5)$
5	$(47, 17, 3)$, $(-52, 17, 3)$
9	$(2, 5, 3)$, $(-11, 5, 3)$
13	$(-11, 5, 3)$, $(-2, 5, 3)$
19	$(2636, 241, 3)$, $(-2655, 241, 3)$
27	$(259, 53, 3)$, $(-286, 53, 3)$
37	$(-46, 13, 3)$, $(9, 13, 3)$
55	$(-9, 13, 3)$, $(-46, 13, 3)$
71	$(137745, 3361, 3)$, $(-137816, 3361, 3)$
73	$(-117, 25, 3)$, $(44, 25, 3)$
77	$(65, 29, 3)$, $(-142, 29, 3)$
79	$(-38, 5, 5)$, $(-41, 5, 5)$
91	$(107, 37, 3)$, $(-198, 37, 3)$
99	$(-47, 17, 3)$, $(-52, 17, 3)$, $(13754, 725, 3)$, $(-13853, 725, 3)$
121	$(-236, 41, 3)$, $(115, 41, 3)$
143	$(478, 85, 3)$, $(-621, 85, 3)$, $(730, 109, 3)$, $(-873, 109, 3)$
161	$(-44, 25, 3)$, $(-117, 25, 3)$
181	$(-415, 61, 3)$, $(234, 61, 3)$
207	$(-142, 29, 3)$, $(-65, 29, 3)$
249	$(29, 5, 7)$, $(-278, 5, 7)$

253	(−666, 85, 3), (413, 85, 3), (296, 73, 3), (−549, 73, 3), (4482, 349, 3), (−4735, 349, 3)
265	(7162792, 46817, 3), (−7163057, 46817, 3)
297	(191, 65, 3), (−488, 65, 3)
305	(−107, 37, 3), (−198, 37, 3)
307	(−29, 5, 7), (−278, 5, 7)
337	(−1001, 113, 3), (664, 113, 3)
351	(−115, 41, 3), (−236, 41, 3)
369	(715957, 10085, 3), (−716326, 10085, 3)
377	(9306, 565, 3), (−9683, 565, 3)
391	(1573, 185, 3), (−1964, 185, 3)
433	(−1432, 145, 3), (999, 145, 3)
475	(5646, 37, 5), (−597, 13, 5), (122, 13, 5), (−6121, 37, 5)
481	(718, 125, 3), (−1199, 125, 3)
517	(7, 65, 3), (−524, 65, 3), (39553, 1469, 3), (−40070, 1469, 3)
531	(−524, 65, 3), (−7, 65, 3)
541	(−1971, 181, 3), (1430, 181, 3)
545	(−286, 53, 3), (−259, 53, 3)
559	(30483, 1237, 3), (−31042, 1237, 3)
585	(803, 137, 3), (−1388, 137, 3)
611	(297, 97, 3), (−908, 97, 3)
629	(1737, 205, 3), (−2366, 205, 3)
649	(−234, 61, 3), (−415, 61, 3)
661	(−2630, 221, 3), (1969, 221, 3)
671	(299, 101, 3), (−970, 101, 3)
679	(−191, 65, 3), (−488, 65, 3)
693	(3404, 305, 3), (−4097, 305, 3)
717	(404, 17, 5), (−1121, 17, 5)
719	(−597, 13, 5), (−122, 13, 5)
747	(−835, 89, 3), (88, 89, 3)
793	(−3421, 265, 3), (2628, 265, 3)
819	(2896, 281, 3), (−3715, 281, 3)
845	(−549, 73, 3), (−296, 73, 3)
851	(18821, 905, 3), (−19672, 905, 3)
923	(−88, 89, 3), (−835, 89, 3)
935	(235639, 4813, 3), (−236574, 4813, 3)
937	(−4356, 313, 3), (3419, 313, 3)
989	(372337471, 652081, 3), (−372338460, 652081, 3)
1035	(1006, 173, 3), (−2041, 173, 3)
1079	(−413, 85, 3), (−666, 85, 3)
1093	(−5447, 365, 3), (4354, 365, 3)
1099	(−478, 85, 3), (−621, 85, 3)
1121	(1694, 221, 3), (−2815, 221, 3)
1205	(−908, 97, 3), (−297, 97, 3)
1207	(828, 169, 3), (−2035, 169, 3)

1261	(−6706, 421, 3), (5445, 421, 3), (431, 145, 3), (−1692, 145, 3)
1269	(−299, 101, 3), (−970, 101, 3)
1287	(−1757, 149, 3), (470, 149, 3), (22552, 1025, 3), (−23839, 1025, 3)
1377	(37219754, 140453, 3), (−37221131, 140453, 3)
1387	(2277, 265, 3), (−3664, 265, 3)
1403	(86256, 2473, 3), (−87659, 2473, 3)
1417	(498157, 7925, 3), (−499574, 7925, 3)
1441	(−8145, 481, 3), (6704, 481, 3)
1457	(10296, 625, 3), (−11753, 625, 3)
1475	(4807, 397, 3), (−6282, 397, 3)
1525	(−1121, 17, 5), (−404, 17, 5)
1603	(−873, 109, 3), (−730, 109, 3)
1611	(−2392, 185, 3), (781, 185, 3)
1633	(−9776, 545, 3), (8143, 545, 3)
1665	(−664, 113, 3), (−1001, 113, 3)
1679	(4268, 377, 3), (−5947, 377, 3)
1819	(143, 157, 3), (−1962, 157, 3)
1837	(−11611, 613, 3), (9774, 613, 3)
1853	(1342, 229, 3), (−3195, 229, 3)
1863	(7135, 509, 3), (−8998, 509, 3)
1891	(11034, 661, 3), (−12925, 661, 3)
1909	(3077, 325, 3), (−4986, 325, 3)
1917	(−718, 125, 3), (−1199, 125, 3)
1925	(2061306, 20413, 3), (−2063231, 20413, 3)
1927	(24992, 1105, 3), (−26919, 1105, 3)
1961	(56881, 1885, 3), (−58842, 1885, 3)
1989	(49480, 1721, 3), (−51469, 1721, 3)
2033	(793, 205, 3), (−2826, 205, 3)
2053	(−13662, 685, 3), (11609, 685, 3)
2093	(1605854, 17285, 3), (−1607947, 17285, 3)
2105	(4449, 13, 7), (−6554, 13, 7), (−1962, 157, 3), (−143, 157, 3)
2115	(587, 197, 3), (−2702, 197, 3)
2123	(−431, 145, 3), (−1692, 145, 3)
2191	(−1388, 137, 3), (−803, 137, 3)
2227	(−470, 149, 3), (−1757, 149, 3)
2281	(−15941, 761, 3), (13660, 761, 3)
2367	(−4070, 269, 3), (1703, 269, 3)
2407	(7414, 533, 3), (−9821, 533, 3)
2431	(−999, 145, 3), (−1432, 145, 3)
2479	(115234, 3005, 3), (−117713, 3005, 3)
2485	(14499, 793, 3), (−16984, 793, 3)
2521	(−18460, 841, 3), (15939, 841, 3)
2645	(−2681, 193, 3), (36, 193, 3)
2673	(203983, 4385, 3), (−206656, 4385, 3)

2717	(−36, 193, 3), (−2681, 193, 3)
2773	(−21231, 925, 3), (18458, 925, 3)
2799	(−5137, 317, 3), (2338, 317, 3)
2807	(−4282, 29, 5), (1475, 29, 5)
2863	(−2035, 169, 3), (−828, 169, 3)
2879	(−3116, 25, 5), (237, 25, 5)
2925	(160067, 3737, 3), (−162992, 3737, 3)
2983	(1726, 293, 3), (−4709, 293, 3)
2989	(44630, 1621, 3), (−47619, 1621, 3)
2997	(5060, 449, 3), (−8057, 449, 3)
3025	(5228, 457, 3), (−8253, 457, 3)
3037	(−24266, 1013, 3), (21229, 1013, 3)
3047	(−1006, 173, 3), (−2041, 173, 3)
3151	(987228, 12505, 3), (−990379, 12505, 3)
3173	(−781, 185, 3), (−2392, 185, 3)
3245	(−3544, 233, 3), (299, 233, 3)
3275	(3186, 373, 3), (−6461, 373, 3)
3281	(767, 257, 3), (−4048, 257, 3)
3289	(−587, 197, 3), (−2702, 197, 3), (1744, 305, 3), (−5033, 305, 3)
3313	(−27577, 1105, 3), (24264, 1105, 3)
3353	(−3116, 25, 5), (−237, 25, 5)
3401	(−1430, 181, 3), (−1971, 181, 3)
3487	(12258250, 66989, 3), (−12261737, 66989, 3)
3509	(8515, 601, 3), (−12024, 601, 3)
3537	(−1964, 185, 3), (−1573, 185, 3)
3601	(−31176, 1201, 3), (27575, 1201, 3)
3619	(−2826, 205, 3), (−793, 205, 3)
3663	(10825, 689, 3), (−14488, 689, 3)
3691	(19354423140, 9082321, 3), (−19354426831, 9082321, 3)
3771	(−7787, 425, 3), (4016, 425, 3)
3827	(2642, 5, 11), (−6469, 5, 11)
3835	(102663, 2797, 3), (−106498, 2797, 3)
3843	(−299, 233, 3), (−3544, 233, 3)
3887	(3573, 409, 3), (−7460, 409, 3)
3901	(−35075, 1301, 3), (31174, 1301, 3)
3905	(−4563, 277, 3), (658, 277, 3), (60931, 1993, 3), (−64836, 1993, 3)
3977	(25624, 1153, 3), (−29601, 1153, 3)
4033	(18557, 949, 3), (−22590, 949, 3)
4103	(−1737, 205, 3), (−2366, 205, 3)
4213	(−39286, 1405, 3), (35073, 1405, 3)
4311	(−9394, 485, 3), (5083, 485, 3)
4347	(95303, 2669, 3), (−99650, 2669, 3)
4393	(495, 289, 3), (−4888, 289, 3)
4433	(7751, 593, 3), (−12184, 593, 3)

4473	(2158, 365, 3), (−6631, 365, 3)
4509	(−2815, 221, 3), (−1694, 221, 3)
4537	(−43821, 1513, 3), (39284, 1513, 3), (−1342, 229, 3), (−3195, 229, 3)
4599	(−1969, 221, 3), (−2630, 221, 3)
4775	(1649, 353, 3), (−6424, 353, 3)
4779	(6284, 545, 3), (−11063, 545, 3)
4807	(971, 325, 3), (−5778, 325, 3)
4815	(−767, 257, 3), (−4048, 257, 3)
4843	(5229, 505, 3), (−10072, 505, 3)
4851	(1221617, 14417, 3), (−1226468, 14417, 3)
4873	(−48692, 1625, 3), (43819, 1625, 3)
4941	(3211, 425, 3), (−8152, 425, 3)
5139	(1934725219, 1956245, 3), (−1934730358, 1956245, 3)
5221	(−53911, 1741, 3), (48690, 1741, 3), (−658, 277, 3), (−4563, 277, 3), (4498091, 34345, 3), (−4503312, 34345, 3)
5243	(41078, 1565, 3), (−46321, 1565, 3)
5251	(267190, 5261, 3), (−272441, 5261, 3)
5291	(−2636, 241, 3), (−2655, 241, 3), (15148, 865, 3), (−20439, 865, 3), (25948522, 110437, 3), (−25953813, 110437, 3)
5311	(551177, 8497, 3), (−556488, 8497, 3)
5383	(−4888, 289, 3), (−495, 289, 3)
5405	(−7117, 377, 3), (1712, 377, 3)
5499	(−13232, 617, 3), (7733, 617, 3)
5581	(−59490, 1861, 3), (53909, 1861, 3)
5611	(10015, 701, 3), (−15626, 701, 3)
5621	(174512, 3977, 3), (−180133, 3977, 3)
5633	(26037, 1189, 3), (−31670, 1189, 3)
5723	(77796, 2353, 3), (−83519, 2353, 3)
5725	(4329, 493, 3), (−10054, 493, 3)
5757	(−1475, 29, 5), (−4282, 29, 5)
5773	(−1703, 269, 3), (−4070, 269, 3)
5941	(−2277, 265, 3), (−3664, 265, 3)
5953	(−65441, 1985, 3), (59488, 1985, 3)
5975	(208, 337, 3), (−6183, 337, 3)
6049	(−2628, 265, 3), (−3421, 265, 3)
6147	(−15487, 689, 3), (9340, 689, 3)
6245	(−8676, 433, 3), (2431, 433, 3)
6265	(11258, 757, 3), (−17523, 757, 3)
6313	(74014, 2285, 3), (−80327, 2285, 3)
6335	(14922, 877, 3), (−21257, 877, 3)
6337	(−71776, 2113, 3), (65439, 2113, 3)
6371	(2638, 445, 3), (−9009, 445, 3)
6391	(−6183, 337, 3), (−208, 337, 3)
6435	(−1726, 293, 3), (−4709, 293, 3)

6557	(184574, 4133, 3), (−191131, 4133, 3)
6611	(−3715, 281, 3), (−2896, 281, 3)
6643	(288629, 5545, 3), (−295272, 5545, 3)
6733	(−78507, 2245, 3), (71774, 2245, 3)
6741	(1199, 401, 3), (−7940, 401, 3)
6749	(−971, 325, 3), (−5778, 325, 3)
6777	(−5033, 305, 3), (−1744, 305, 3)
6903	(21161, 1073, 3), (−28064, 1073, 3)
6931	(3140, 481, 3), (−10071, 481, 3)
6989	(7436, 641, 3), (−14425, 641, 3)
7037	(5302, 565, 3), (−12339, 565, 3)
7097	(1319591, 15185, 3), (−1326688, 15185, 3)
7141	(−85646, 2381, 3), (78505, 2381, 3)
7145	(−10439, 493, 3), (3294, 493, 3)
7183	(107167895, 284269, 3), (−107175078, 284269, 3)
7191	(49430, 1781, 3), (−56621, 1781, 3)
7245	(444026, 7373, 3), (−451271, 7373, 3)
7259	(647075, 9461, 3), (−654334, 9461, 3)
7267	(28888, 1289, 3), (−36155, 1289, 3)
7339	(20491, 1061, 3), (−27830, 1061, 3)
7363	(3031686, 26413, 3), (−3039049, 26413, 3)
7371	(2592749, 23801, 3), (−2600120, 23801, 3)
7379	(38259, 1525, 3), (−45638, 1525, 3)
7475	(−2338, 317, 3), (−5137, 317, 3)
7483	(−7670, 389, 3), (187, 389, 3)
7501	(−3404, 305, 3), (−4097, 305, 3)
7551	(−20729, 845, 3), (13178, 845, 3)
7561	(−93205, 2521, 3), (85644, 2521, 3)
7579	(51209, 1825, 3), (−58788, 1825, 3)
7775	(−3419, 313, 3), (−4356, 313, 3)
7813	(83553173, 240805, 3), (−83560986, 240805, 3)
7847	(19841, 65, 5), (−27688, 65, 5)
7849	(33588, 1417, 3), (−41437, 1417, 3)
7857	(−187, 389, 3), (−7670, 389, 3)
7957	(148005, 3589, 3), (−155962, 3589, 3)
7993	(−101196, 2665, 3), (93203, 2665, 3)
7999	(−10475, 41, 5), (2476, 41, 5)
8063	(−4986, 325, 3), (−3077, 325, 3)
8073	(−6424, 353, 3), (−1649, 353, 3)
8105	(−12418, 557, 3), (4313, 557, 3)
8217	(11635, 809, 3), (−19852, 809, 3)
8307	(−23740, 929, 3), (15433, 929, 3)
8437	(−109631, 2813, 3), (101194, 2813, 3)
8541	(15689, 941, 3), (−24230, 941, 3)

8549	(413829, 7045, 3), (−422378, 7045, 3)
8659	(−9361, 445, 3), (702, 445, 3)
8671	(1159, 461, 3), (−9830, 461, 3)
8725	(3166, 533, 3), (−11891, 533, 3)
8789	(−2158, 365, 3), (−6631, 365, 3)
8829	(−1712, 377, 3), (−7117, 377, 3)
8893	(−118522, 2965, 3), (109629, 2965, 3)
9017	(781300, 10729, 3), (−790317, 10729, 3)
9111	(−6469, 5, 11), (−2642, 5, 11)
9131	(1451, 485, 3), (−10582, 485, 3)
9139	(−1199, 401, 3), (−7940, 401, 3)
9217	(−4735, 349, 3), (−4482, 349, 3)
9269	(2961, 541, 3), (−12230, 541, 3)
9287	(11583, 829, 3), (−20870, 829, 3), (6084559, 42013, 3), (−6093846, 42013, 3)
9361	(−127881, 3121, 3), (118520, 3121, 3), (17082, 997, 3), (−26443, 997, 3)
9603	(5267, 629, 3), (−14870, 629, 3)
9647	(−6461, 373, 3), (−3186, 373, 3)
9703	(8684, 745, 3), (−18387, 745, 3)
9729	(124306, 3221, 3), (−134035, 3221, 3)
9801	(−4354, 365, 3), (−5447, 365, 3)
9841	(−137720, 3281, 3), (127879, 3281, 3)
9855	(26962, 1277, 3), (−36817, 1277, 3)
9919	(−11268, 505, 3), (1349, 505, 3)
9927	(−30602, 1109, 3), (20675, 1109, 3)
9999	(10705088, 61217, 3), (−10715087, 61217, 3)

8B. Triples of nontrivial solutions $(x, |y|, n)$ of (1) for $d = 2$, $n = 4$ and for $1 \le r \le 10^4$.

r	$(x, \|y\|, n)$
1	(118, 13, 4), (−121, 13, 4), (−1, 1, 4), (−2, 1, 4)
17	(−10, 5, 4), (−41, 5, 4)
31	(−55, 5, 4), (−38, 5, 4)
79	(−319, 17, 4), (82, 17, 4)
191	(−718, 25, 4), (145, 25, 4)
239	(−359, 13, 4), (−358, 13, 4)
241	(599, 37, 4), (−1322, 37, 4)
401	(−562, 17, 4), (−641, 17, 4)
799	(−79, 41, 4), (−758, 29, 4), (−1639, 29, 4), (−2318, 41, 4)
863	(−1199, 25, 4), (−1390, 25, 4)
881	(−1721, 29, 4), (−922, 29, 4)
911	(−6455, 85, 4), (3722, 85, 4)
1279	(−38, 53, 4), (−3799, 53, 4)
1361	(−7601, 89, 4), (3518, 89, 4)
1457	(8839, 125, 4), (−13210, 125, 4)

1649	(398, 65, 4), (−5345, 65, 4)
1697	(319, 65, 4), (−5410, 65, 4)
1921	(−2761, 37, 4), (−3002, 37, 4)
2159	(−839, 61, 4), (−5638, 61, 4)
2239	(−2959, 41, 4), (−3758, 41, 4)
2719	(−19718, 149, 4), (11561, 149, 4)
3503	(24410, 205, 4), (−34919, 205, 4)
3761	(−5002, 53, 4), (−6281, 53, 4)
4369	(43055, 265, 4), (−56162, 265, 4)
4559	(−9839, 73, 4), (−3838, 73, 4)
4703	(−2519, 85, 4), (−11590, 85, 4)
4799	(−8278, 61, 4), (−6119, 61, 4)
5441	(−14842, 101, 4), (−1481, 101, 4)
5729	(−9442, 65, 4), (−7745, 65, 4)
5743	(−9439, 65, 4), (−7790, 65, 4)
6001	(−11281, 73, 4), (−6722, 73, 4)
6239	(−1558, 109, 4), (−17159, 109, 4)
7361	(9799, 173, 4), (−31882, 173, 4)
7663	(−35390, 185, 4), (12401, 185, 4)
7681	(42598, 277, 4), (−65641, 277, 4)
8401	(−17761, 97, 4), (−7442, 97, 4)
8959	(−113839, 377, 4), (86962, 377, 4), (−5599, 113, 4), (−21278, 113, 4)
9071	(−11255, 85, 4), (−15958, 85, 4)
9601	(133199, 457, 4), (−162002, 457, 4)

8C. Triples of nontrivial solutions (x, y, n) of (1) for $d = 6$ and prime $n \geq 3$ for $1 \leq r \leq 10^4$.

r	(x, y, n)
13	(−20, 19, 3), (−71, 19, 3)
23	(−22, 31, 3), (−139, 31, 3)
55	(−828, 19, 5), (443, 19, 5)
347	(−1525, 139, 3), (−904, 139, 3)
365	(4082, 559, 3), (−6637, 559, 3)
455	(1970807, 28579, 3), (−1973992, 28579, 3)
527	(−2554, 199, 3), (−1135, 199, 3)
535	(4348, 619, 3), (−8093, 619, 3)
679	(12697, 1111, 3), (−17450, 1111, 3)
743	(−3907, 271, 3), (−1294, 271, 3)
851	(2328605, 31951, 3), (−2334562, 31951, 3)
1145	(−3034, 31, 5), (−4981, 31, 5)
1283	(−7729, 451, 3), (−1252, 451, 3)
1391	(56362832, 267139, 3), (−56372569, 267139, 3)
1607	(−10270, 559, 3), (−979, 559, 3)
1615	(1231, 691, 3), (−12536, 691, 3)

1985	$(-4999, 451, 3), (-8896, 451, 3)$
2165	$(-6922, 439, 3), (-8233, 439, 3)$
2191	$(5482, 1039, 3), (-20819, 1039, 3)$
2263	$(1360645, 22399, 3), (-1376486, 22399, 3)$
2363	$(-16792, 811, 3), (251, 811, 3)$
2669	$(214052, 6691, 3), (-232735, 6691, 3)$
2813	$(1109606, 19591, 3), (-1129297, 19591, 3)$
2893	$(53803, 2911, 3), (-74054, 2911, 3)$
2933	$(865, 19, 7), (-21396, 19, 7)$
2983	$(302191, 8371, 3), (-323072, 8371, 3)$
3101	$(7328, 1291, 3), (-29035, 1291, 3)$
3263	$(-25474, 1111, 3), (2633, 1111, 3)$
3451	$(-1049, 979, 3), (-23108, 979, 3)$
3767	$(-30715, 1279, 3), (4346, 1279, 3)$
4117	$(263895274, 747631, 3), (-263924093, 747631, 3)$
4199	$(90320, 4051, 3), (-119713, 4051, 3)$
4307	$(-36604, 1459, 3), (6455, 1459, 3)$
4315	$(-7631, 871, 3), (-22574, 871, 3)$
4387	$(3160291, 39259, 3), (-3191000, 39259, 3)$
4883	$(-43177, 1651, 3), (8996, 1651, 3)$
5369	$(503, 1399, 3), (-38086, 1399, 3)$
5423	$(36224, 2659, 3), (-74185, 2659, 3)$
5719	$(-16178, 871, 3), (-23855, 871, 3)$
5935	$(-13448, 979, 3), (-28097, 979, 3)$
5971	$(-19613, 859, 3), (-22184, 859, 3)$
6143	$(-58519, 2071, 3), (15518, 2071, 3)$
6827	$(-67360, 2299, 3), (19571, 2299, 3)$
7501	$(66655, 3751, 3), (-119162, 3751, 3)$
7547	$(-77029, 2539, 3), (24200, 2539, 3)$
8303	$(-87562, 2791, 3), (29441, 2791, 3)$
8987	$(18857, 2551, 3), (-81766, 2551, 3)$
9715	$(-28034, 1231, 3), (-39971, 1231, 3)$
9923	$(-111364, 3331, 3), (41903, 3331, 3)$

Acknowledgement

Both authors thank the University of Toronto and their mathematics department for their support. We also thank Szabolcs Tengely for his comments on a first draft of this manuscript and for pointing towards work that helped us to amend some proofs and greatly increase the efficiency and speed of computations.

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DEBANJANA KUNDU: dkundu@math.utoronto.ca
Centre de recherches mathematiques, Montreal, Canada

VANDITA PATEL: vandita.patel@manchester.ac.uk
Department of Mathematics, The University of Manchester, Manchester, United Kingdom

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Rocky Mountain Mathematics Consortium